

# MULTI-SUBLINEAR OPERATORS GENERATED BY MULTILINEAR FRACTIONAL INTEGRAL OPERATORS AND COMMUTATORS ON THE PRODUCT GENERALIZED LOCAL MORREY SPACES

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**ABSTRACT.** The aim of this paper is to get the boundedness of certain multi-sublinear operators generated by multilinear fractional integral operators on the product generalized local Morrey spaces under generic size conditions which are satisfied by most of the operators in harmonic analysis. We also prove that the commutators of multilinear operators generated by local Campanato functions and multilinear fractional integral operators are also bounded on the product generalized local Morrey spaces.

## 1. INTRODUCTION

The classical Morrey spaces  $L_{p,\lambda}$  were introduced by Morrey [41] in 1938 to study the local behavior of solutions of second order elliptic partial differential equations (PDEs). Later, there were many applications of Morrey space to the Navier-Stokes equations (see [38]), the Schrödinger equations (see [46]) and the elliptic problems with discontinuous coefficients (see [5, 43]).

Let  $\mathbb{R}^n$  be the  $n$ -dimensional Euclidean space of points  $x = (x_1, \dots, x_n)$  with norm  $|x| = (\sum_{i=1}^n x_i^2)^{1/2}$  and corresponding  $m$ -fold product spaces ( $m \in \mathbb{N}$ ) be  $(\mathbb{R}^n)^m = \mathbb{R}^n \times \dots \times \mathbb{R}^n$ . Let  $B = B(x, r)$  denotes open ball centered at  $x$  of radius  $r$  for  $x \in \mathbb{R}^n$  and  $r > 0$  and  $B^c(x, r)$  its complement. Also  $|B(x, r)|$  is the Lebesgue measure of the ball  $B(x, r)$  and  $|B(x, r)| = v_n r^n$ , where  $v_n = |B(0, 1)|$ . For a given measurable set  $E$ , we also denote the Lebesgue measure of  $E$  by  $|E|$ . For any given  $X \subseteq \mathbb{R}^n$  and  $0 < p < \infty$ , denote by  $L_p(X)$  the spaces of all measurable functions  $f$  satisfying

$$\|f\|_{L_p(X)} = \left( \int_X |f(x)|^p dx \right)^{1/p} < \infty.$$

We recall the definition of classical Morrey spaces  $L_{p,\lambda}$  as

$$L_{p,\lambda}(\mathbb{R}^n) = \left\{ f : \|f\|_{L_{p,\lambda}(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n, r > 0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))} < \infty \right\},$$

where  $f \in L_p^{loc}(\mathbb{R}^n)$ ,  $0 \leq \lambda \leq n$  and  $1 \leq p < \infty$ .

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Note that  $L_{p,0} = L_p(\mathbb{R}^n)$  and  $L_{p,n} = L_\infty(\mathbb{R}^n)$ . If  $\lambda < 0$  or  $\lambda > n$ , then  $L_{p,\lambda} = \Theta$ , where  $\Theta$  is the set of all functions equivalent to 0 on  $\mathbb{R}^n$ . It is known that  $L_{p,\lambda}(\mathbb{R}^n)$  is an extension of  $L_p(\mathbb{R}^n)$  in the sense that  $L_{p,0} = L_p(\mathbb{R}^n)$ .

We also denote by  $WL_{p,\lambda} \equiv WL_{p,\lambda}(\mathbb{R}^n)$  the weak Morrey space of all functions  $f \in WL_p^{loc}(\mathbb{R}^n)$  for which

$$\|f\|_{WL_{p,\lambda}} \equiv \|f\|_{WL_{p,\lambda}(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n, r > 0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,r))} < \infty,$$

where  $WL_p(B(x,r))$  denotes the weak  $L_p$ -space of measurable functions  $f$  for which

$$\|f\|_{WL_p(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{WL_p(\mathbb{R}^n)} = \sup_{t>0} t |\{y \in B(x,r) : |f(y)| > t\}|^{1/p}.$$

For the boundedness of the Hardy–Littlewood maximal operator, the fractional integral operator and the Calderón–Zygmund singular integral operator on Morrey spaces, we refer the readers to [1, 8, 45]. For the properties and applications of classical Morrey spaces, see [9, 10, 27] and the references therein.

Let  $f \in L_1^{loc}(\mathbb{R}^n)$ . The Hardy–Littlewood maximal operator  $M$  is defined by

$$Mf(x) = \sup_{t>0} |B(x,t)|^{-1} \int_{B(x,t)} |f(y)| dy.$$

It's well known that  $M$  is bounded on  $L_p(\mathbb{R}^n)$  for  $1 < p \leq \infty$  and for  $p = 1$  weak type inequality also holds.

It is well known that the standard Calderón–Zygmund singular integral operator, briefly a Calderón–Zygmund operator  $\bar{T}$  has the following integral expression

$$\bar{T}f(x) = \int_{\mathbb{R}^n} K(x,y) f(y) dy,$$

for any test function  $f$  and  $x \notin \text{supp} f$ . Here  $K$  is the Calderón–Zygmund kernel, which is a locally integrable function defined away from the diagonal and satisfies the following size condition:

$$|K(x,y)| \leq C |x-y|^{-n}, \quad x \neq y,$$

and some continuity assumptions. Boundedness of Calderón–Zygmund operator  $\bar{T}$  on  $L_p(\mathbb{R}^n)$  for any  $1 < p < \infty$  is well known.

Let  $f \in L^{loc}(\mathbb{R}^n)$ . The fractional maximal operator  $M_\alpha$  and the fractional integral operator  $\bar{T}_\alpha$  (also known as the Riesz potential  $I_\alpha$ ) are defined respectively by

$$M_\alpha f(x) = \sup_{t>0} |B(x,t)|^{-1+\frac{\alpha}{n}} \int_{B(x,t)} |f(y)| dy, \quad 0 \leq \alpha < n$$

$$\bar{T}_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \quad 0 < \alpha < n.$$

$M_\alpha$  and  $\bar{T}_\alpha$  play important roles in harmonic analysis (see [36, 53, 55]). Also, the fractional integral play an essential role in PDEs. It is well known that, see [53] for example,  $\bar{T}_\alpha$  is bounded from  $L_p(\mathbb{R}^n)$  to  $L_q(\mathbb{R}^n)$  for all  $p > 1$  and  $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n} > 0$ , and  $\bar{T}_\alpha$  is also of weak type  $\left(1, \frac{n}{n-\alpha}\right)$ , this is known as Hardy–Littlewood

Sobolev inequality. Boundedness of the fractional integral operator  $\overline{T}_\alpha$  on the space  $M_{p,\lambda}(\mathbb{R}^n)$  has been studied by Spanne (published by Peetre [45]) and Adams [1].

Recall that, for  $0 < \alpha < n$ ,

$$M_\alpha f(x) \leq \nu_n^{\frac{\alpha}{n}-1} \overline{T}_\alpha(|f|)(x)$$

holds (see [31], Remark 2.1). Hence one gets the boundedness of the fractional maximal operator  $M_\alpha$  from the boundedness of  $\overline{T}_\alpha$ , where  $\nu_n$  is the volume of the unit ball on  $\mathbb{R}^n$ .

In 1976, Coifman et al. [12] introduced the commutator  $\overline{T}_b$  generated by the Calderón-Zygmund operator  $\overline{T}$  and a locally integrable function  $b$  as follows:

$$(1.1) \quad \overline{T}_b f(x) \equiv b(x)\overline{T}f(x) - \overline{T}(bf)(x) = \int_{\mathbb{R}^n} K(x, y) [b(x) - b(y)] f(y) dy.$$

It is well known from [12] that  $\overline{T}_b$  is a bounded operator on  $L_p(\mathbb{R}^n)$ ,  $1 < p < \infty$  if and only if  $b \in BMO$  (bounded mean oscillation).

Let  $b$  be a locally integrable function on  $\mathbb{R}^n$ . Then we shall define the commutators for a suitable function  $f$  generated by the fractional integral operators and  $b$  as follows:

$$[b, \overline{T}_\alpha]f(x) \equiv b(x)\overline{T}_\alpha f(x) - \overline{T}_\alpha(bf)(x) = \int_{\mathbb{R}^n} [b(x) - b(y)] \frac{f(y)}{|x - y|^{n-\alpha}} dy,$$

where  $0 < \alpha < n$ .

We denote by  $\vec{y} = (y_1, \dots, y_m)$ ,  $d\vec{y} = dy_1 \dots dy_m$ , and by  $\vec{f}$  the  $m$ -tuple  $(f_1, \dots, f_m)$ ,  $m, n$  the nonnegative integers with  $n \geq 2$ ,  $m \geq 1$ .

Let  $\vec{f} \in L_{p_1}^{loc}(\mathbb{R}^n) \times \dots \times L_{p_m}^{loc}(\mathbb{R}^n)$ . Then multi-sublinear fractional maximal operator  $M_\alpha^{(m)}$  is defined by

$$M_\alpha^{(m)}(\vec{f})(x) = \sup_{t>0} |B(x, t)|^{\frac{\alpha}{n}} \left[ \prod_{i=1}^m \frac{1}{|B(x, t)|} \int_{B(x, t)} |f_i(y_i)| dy_i \right], \quad 0 \leq \alpha < mn.$$

From definition, if  $\alpha = 0$  then  $M_\alpha^{(m)}$  is the multi-sublinear maximal operator  $M^{(m)}$  and also; in the case of  $m = 1$ ,  $M_\alpha^{(m)}$  is the classical fractional maximal operator  $M_\alpha$ .

After the work of Coifman and Meyer [11] the multilinear theory is received increasing attention. Multilinear Calderón-Zygmund operators are studied by Grafakos-Torres [20, 21, 22] and Grafakos-Kalton [18] and the multilinear fractional integral operators by Grafakos [17] and Kenig-Stein [30].

Firstly, recall that the  $m$ (multi)-linear Calderón-Zygmund operator  $\overline{T}^{(m)}$  ( $m \in \mathbb{N}$ ) for test vector  $\vec{f} = (f_1, \dots, f_m)$  is defined by

$$\overline{T}^{(m)}(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \dots, y_m) \left\{ \prod_{i=1}^m f_i(y_i) \right\} dy_1 \dots dy_m, \quad x \notin \bigcap_{i=1}^m \text{supp} f_i,$$

where  $K$  is an  $m$ -Calderón-Zygmund kernel which is a locally integrable function defined away from the diagonal  $y_0 = y_1 = \dots = y_m$  on  $(\mathbb{R}^n)^{m+1}$  satisfying the

following size estimate:

$$(1.2) \quad |K(x, y_1, \dots, y_m)| \leq \frac{C}{|(x - y_1, \dots, x - y_m)|^{mn}},$$

for some  $C > 0$  and some smoothness estimates, see [20-22] for details.

The result of Grafakos and Torres [20, 22] shows that the multilinear Calderón-Zygmund operator is bounded on the product of Lebesgue spaces.

**Theorem 1.** [20, 22] *Let  $\overline{T}^{(m)}$  ( $m \in \mathbb{N}$ ) is a  $m$ -linear Calderón-Zygmund operator. Then, for any numbers  $1 \leq p_1, \dots, p_m < \infty$  with  $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$ ,  $\overline{T}^{(m)}$  can be extended to a bounded operator from  $L_{p_1} \times \dots \times L_{p_m}$  into  $L_p$ , and bounded from  $L_1 \times \dots \times L_1$  into  $L_{\frac{1}{m}, \infty}$ .*

In this paper we deal with another kind of multilinear operator for  $\vec{f} = (f_1, \dots, f_m)$ , which is called multilinear fractional integral operator as follows

$$\overline{T}_\alpha^{(m)}(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} \frac{1}{|(x - y_1, \dots, x - y_m)|^{mn-\alpha}} \left\{ \prod_{i=1}^m f_i(y_i) \right\} d\vec{y},$$

whose kernel is

$$(1.3) \quad |K(x, y_1, \dots, y_m)| = |(x - y_1, \dots, x - y_m)|^{-mn+\alpha}, \quad 0 < \alpha < mn,$$

where  $f_1, \dots, f_m : \mathbb{R}^n \rightarrow \mathbb{R}$  are measurable and  $|(x - y_1, \dots, x - y_m)| = \sqrt{\sum_{i=1}^m |x - y_i|^2}$ .

It is well known that multilinear fractional integral operator was first studied by Grafakos [17]. In the following result Kenig and Stein [30] have proved that the multilinear fractional integral operator is bounded on the product of Lebesgue spaces.

**Theorem 2.** [30] *Let  $0 < \alpha < mn$ ,  $\overline{T}_\alpha^{(m)}$  ( $m \in \mathbb{N}$ ) be an  $m$ -linear fractional integral operator with kernel  $K$  satisfying (1.3) and  $f_i \in L_{p_i}(\mathbb{R}^n)$  ( $i = 1, \dots, m$ ) with  $1 \leq p_i \leq \infty$  and  $\frac{1}{q} = \frac{1}{p_1} + \dots + \frac{1}{p_m} - \frac{\alpha}{n} > 0$ .*

(1) *If each  $p_i > 1$ , then*

$$\left\| \overline{T}_\alpha^{(m)}(\vec{f}) \right\|_{L_q(\mathbb{R}^n)} \leq C \prod_{i=1}^m \|f_i\|_{L_{p_i}(\mathbb{R}^n)};$$

(2) *If  $p_i = 1$  for some  $i$ , then*

$$\left\| \overline{T}_\alpha^{(m)}(\vec{f}) \right\|_{L_{q, \infty}(\mathbb{R}^n)} \leq C \prod_{i=1}^m \|f_i\|_{L_{p_i}(\mathbb{R}^n)},$$

here  $L_{q, \infty}(\mathbb{R}^n)$  denotes the weak  $L_q(\mathbb{R}^n)$  space, the constant  $C > 0$  independent of  $\vec{f}$ .

If we take  $m = 1$ ,  $\overline{T}_\alpha^{(m)}$  ( $m \in \mathbb{N}$ ) is the classical fractional integral operator  $\overline{T}_\alpha$ . Moreover, Theorem 2 is the multi-version of well-known Hardy-Littlewood-Sobolev inequality. Weighted inequalities for the multilinear fractional integral operators have been established by Moen [40] and Chen and Xue [7].

Xu [58] has established the boundedness of the commutators generated by  $m$ -linear Calderón-Zygmund singular integrals and  $RBMO$  functions with nonhomogeneity on the product of Lebesgue space. Inspired by [20], [22], [58], commutators  $\overline{T}_{\vec{b}}^{(m)}$  generated by  $m$ -linear Calderón-Zygmund operators  $\overline{T}^{(m)}$  and local Campanato functions  $\vec{b} = (b_1, \dots, b_m)$  is given by

$$\overline{T}_{\vec{b}}^{(m)}(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \dots, y_m) \left[ \prod_{i=1}^m [b_i(x) - b_i(y_i)] f_i(y_i) \right] d\vec{y},$$

where  $K(x, y_1, \dots, y_m)$  is a  $m$ -linear Calderón-Zygmund kernel,  $b_i \in LC_{q_i, \lambda_i}^{\{x_0\}}(\mathbb{R}^n)$  (local Campanato spaces, see for definition in Section 4) for  $0 \leq \lambda_i < \frac{1}{n}$ ,  $i = 1, \dots, m$ . Note that  $\overline{T}_b$  is the special case of  $\overline{T}_{\vec{b}}^{(m)}$  with taking  $m = 1$ . Similarly, let  $b_i$  ( $i = 1, \dots, m$ ) be a locally integrable functions on  $\mathbb{R}^n$ , then the commutators generated by  $m$ -linear fractional integral operators and  $\vec{b} = (b_1, \dots, b_m)$  is given by

$$\overline{T}_{\alpha, \vec{b}}^{(m)}(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} \frac{1}{|(x - y_1, \dots, x - y_m)|^{mn-\alpha}} \left[ \prod_{i=1}^m [b_i(x) - b_i(y_i)] f_i(y_i) \right] d\vec{y},$$

where  $0 < \alpha < mn$ , and  $f_i$  ( $i = 1, \dots, m$ ) are suitable functions.

Suppose that  $T_{\alpha}^{(m)}$ ,  $\alpha \in (0, mn)$  represents a multilinear or a multi-sublinear operator, which satisfies that for any  $m \in \mathbb{N}$ ,  $\vec{f} = (f_1, \dots, f_m)$ , and  $x \notin \bigcap_{i=1}^m \text{supp} f_i$ ,

$$(1.4) \quad \left| T_{\alpha}^{(m)}(\vec{f})(x) \right| \leq c_0 \int_{(\mathbb{R}^n)^m} \frac{1}{|(x - y_1, \dots, x - y_m)|^{mn-\alpha}} \left\{ \prod_{i=1}^m |f_i(y_i)| \right\} d\vec{y},$$

where  $c_0$  is independent of  $\vec{f}$ ,  $x$  and each  $f_i$  ( $i = 1, \dots, m$ ) is integrable on  $\mathbb{R}^n$  with compact support.

Condition (1.4) in the case of  $m = 1$  was first introduced by Soria and Weiss in [52] and is satisfied by many interesting operators in harmonic analysis, such as the  $m$ -linear fractional integral operator, multi-sublinear fractional maximal operator and so on (see [35], [50], [52] for details).

In this paper we will establish the boundedness of a large class of multi-sublinear operators, including multi-sublinear fractional maximal operators, with multilinear fractional integral operators as their special cases, and give local Campanato space estimates for commutators on the product generalized local Morrey spaces.

At last, throughout the paper we use the letter  $C$  for a positive constant, independent of appropriate parameters and not necessarily the same at each occurrence. By  $A \lesssim B$  we mean that  $A \leq CB$  with some positive constant  $C$  independent of appropriate quantities. If  $A \lesssim B$  and  $B \lesssim A$ , we write  $A \approx B$  and say that  $A$  and  $B$  are equivalent.

## 2. GENERALIZED LOCAL MORREY SPACES

After studying Morrey spaces in detail, researchers have passed to generalized Morrey spaces. Mizuhara [39] has given generalized Morrey spaces  $M_{p, \varphi}$  considering  $\varphi = \varphi(r)$  instead of  $r^\lambda$  in the above definition of the Morrey space. Later,

Guliyev [23] has defined the generalized Morrey spaces  $M_{p,\varphi}$  with normalized norm as follows:

**Definition 1.** [23] Let  $\varphi(x, r)$  be a positive measurable function on  $\mathbb{R}^n \times (0, \infty)$  and  $1 \leq p < \infty$ .  $M_{p,\varphi} \equiv M_{p,\varphi}(\mathbb{R}^n)$  denotes the generalized Morrey space, the space of all functions  $f \in L_p^{loc}(\mathbb{R}^n)$  with finite quasinorm

$$\|f\|_{M_{p,\varphi}} = \sup_{x \in \mathbb{R}^n, r > 0} \varphi(x, r)^{-1} |B(x, r)|^{-\frac{1}{p}} \|f\|_{L_p(B(x, r))} < \infty.$$

Also  $WM_{p,\varphi} \equiv WM_{p,\varphi}(\mathbb{R}^n)$  denotes the weak generalized Morrey space of all functions  $f \in WL_p^{loc}(\mathbb{R}^n)$  for which

$$\|f\|_{WM_{p,\varphi}} = \sup_{x \in \mathbb{R}^n, r > 0} \varphi(x, r)^{-1} |B(x, r)|^{-\frac{1}{p}} \|f\|_{WL_p(B(x, r))} < \infty.$$

Everywhere in the sequel we assume that  $\inf_{x \in \mathbb{R}^n, r > 0} \varphi(x, r) > 0$  which makes the above spaces non-trivial, since the spaces of bounded functions are contained in these spaces.

In [23], [28], [32], [39], [42] and [51], the boundedness of the maximal operator and Calderón-Zygmund operator on the generalized Morrey spaces has been obtained, respectively. For generalized Morrey spaces with nondoubling measures see also [48].

There are many papers discussing the conditions on  $\varphi(x, r)$  to obtain the boundedness of operators on the generalized Morrey spaces. For example, in [42], the following condition has been imposed on  $\varphi(x, r)$ :

$$(2.1) \quad c^{-1} \varphi(x, r) \leq \varphi(x, t) \leq c \varphi(x, r),$$

whenever  $r \leq t \leq 2r$ , where  $c (\geq 1)$  does not depend on  $t, r$  and  $x \in \mathbb{R}^n$ , jointly with the condition:

$$(2.2) \quad \int_r^\infty \varphi(x, t)^p \frac{dt}{t} \leq C \varphi(x, r)^p,$$

for some operators  $T$  satisfying the condition (1.4) (by taking  $m = 1$  there), where  $C$  does not depend on  $x \in \mathbb{R}^n$  and  $r$ .

In [14] the boundedness of sublinear operators satisfying condition (1.4) (by taking  $m = 1$  there) has been proved. For the properties of generalized Morrey spaces  $M_{p,\varphi}$ , see also [23, 28, 51, ?].

Recall that the concept of the generalized local (central) Morrey space  $LM_{p,\varphi}^{\{x_0\}}$  has been introduced in [4] and studied in [24, 25, 26].

**Definition 2.** Let  $\varphi(x, r)$  be a positive measurable function on  $\mathbb{R}^n \times (0, \infty)$  and  $1 \leq p < \infty$ . For any fixed  $x_0 \in \mathbb{R}^n$  we denote by  $LM_{p,\varphi}^{\{x_0\}} \equiv LM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$  the generalized local Morrey space, the space of all functions  $f \in L_p^{loc}(\mathbb{R}^n)$  with finite quasinorm

$$\|f\|_{LM_{p,\varphi}^{\{x_0\}}} = \sup_{r > 0} \varphi(x_0, r)^{-1} |B(x_0, r)|^{-\frac{1}{p}} \|f\|_{L_p(B(x_0, r))} < \infty.$$

Also by  $WLM_{p,\varphi}^{\{x_0\}} \equiv WLM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$  the weak generalized local Morrey space of all functions  $f \in WL_p^{loc}(\mathbb{R}^n)$  for which

$$\|f\|_{WLM_{p,\varphi}^{\{x_0\}}} = \sup_{r > 0} \varphi(x_0, r)^{-1} |B(x_0, r)|^{-\frac{1}{p}} \|f\|_{WL_p(B(x_0, r))} < \infty.$$

As in the above, everywhere in the sequel we assume that  $\inf_{r>0} \varphi(x_0, r) > 0$  for the same reasons.

According to this definition, we recover the local Morrey space  $LM_{p,\lambda}^{\{x_0\}}$  and the weak local Morrey space  $WLM_{p,\lambda}^{\{x_0\}}$  under the choice  $\varphi(x_0, r) = r^{\frac{\lambda-n}{p}}$ :

$$LM_{p,\lambda}^{\{x_0\}} = LM_{p,\varphi}^{\{x_0\}} \big|_{\varphi(x_0,r)=r^{\frac{\lambda-n}{p}}}, \quad WLM_{p,\lambda}^{\{x_0\}} = WLM_{p,\varphi}^{\{x_0\}} \big|_{\varphi(x_0,r)=r^{\frac{\lambda-n}{p}}}.$$

The main goal of [4, 24, 25, 26] is to give some sufficient conditions for the boundedness of a large class of rough sublinear operators and their commutators on the generalized local Morrey space  $LM_{p,\varphi}^{\{x_0\}}$ . For the properties and applications of generalized local Morrey spaces  $LM_{p,\varphi}^{\{x_0\}}$ , see [4, 24, 25, 26].

Furthermore, we have the following embeddings:

$$M_{p,\varphi} \subset LM_{p,\varphi}^{\{x_0\}}, \quad \|f\|_{LM_{p,\varphi}^{\{x_0\}}} \lesssim \|f\|_{M_{p,\varphi}},$$

$$WM_{p,\varphi} \subset WLM_{p,\varphi}^{\{x_0\}}, \quad \|f\|_{WLM_{p,\varphi}^{\{x_0\}}} \lesssim \|f\|_{WM_{p,\varphi}}.$$

Now, we can give  $\lambda$ -central bounded mean oscillation space's historical development.

Wiener [56, 57] has looked for a way to describe the behavior of a function at the infinity. The conditions which he has considered are related to appropriate weighted  $L_q$  spaces. Beurling [3] has extended this idea and has defined a pair of dual Banach spaces  $A_q$  and  $B_{q'}$ , where  $1/q + 1/q' = 1$ . To be precise,  $A_q$  is a Banach algebra with respect to the convolution, expressed as a union of certain weighted  $L_q$  spaces; the space  $B_{q'}$  is expressed as the intersection of the corresponding weighted  $L_{q'}$  spaces. Feichtinger [15] has observed that the space  $B_q$  can be described by

$$(2.3) \quad \|f\|_{B_q} = \sup_{k \geq 0} 2^{-\frac{kn}{q}} \|f\chi_k\|_{L_q(\mathbb{R}^n)} < \infty,$$

where  $\chi_0$  is the characteristic function of the unit ball  $\{x \in \mathbb{R}^n : |x| \leq 1\}$ ,  $\chi_k$  is the characteristic function of the annulus  $\{x \in \mathbb{R}^n : 2^{k-1} < |x| \leq 2^k\}$ ,  $k = 1, 2, \dots$ . By duality, the space  $A_q(\mathbb{R}^n)$ , appropriately called now the Beurling algebra, can be described by the condition

$$(2.4) \quad \|f\|_{A_q} = \sum_{k=0}^{\infty} 2^{-\frac{kn}{q'}} \|f\chi_k\|_{L_q(\mathbb{R}^n)} < \infty.$$

Let  $\dot{B}_q(\mathbb{R}^n)$  and  $\dot{A}_q(\mathbb{R}^n)$  be the homogeneous versions of  $B_q(\mathbb{R}^n)$  and  $A_q(\mathbb{R}^n)$  by taking  $k \in \mathbb{Z}$  in (2.3) and (2.4) instead of  $k \geq 0$  there (see [16] for details).

If  $\lambda < 0$  or  $\lambda > n$ , then  $LM_{p,\lambda}^{\{x_0\}}(\mathbb{R}^n) = \Theta$ . Note that  $LM_{p,0}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$  and  $LM_{p,n}(\mathbb{R}^n) = \dot{B}_p(\mathbb{R}^n)$ .

$$\dot{B}_{p,\mu} = LM_{p,\varphi} \big|_{\varphi(0,r)=r^{\mu n}}, \quad W\dot{B}_{p,\mu} = WLM_{p,\varphi} \big|_{\varphi(0,r)=r^{\mu n}}.$$

Alvarez et al. [2], in order to study the relationship between central  $BMO$  spaces and Morrey spaces, introduced  $\lambda$ -central bounded mean oscillation spaces and central Morrey spaces  $\dot{B}_{p,\mu}(\mathbb{R}^n) \equiv LM_{p,n+n\mu}(\mathbb{R}^n)$ ,  $\mu \in [-\frac{1}{p}, 0]$ . If  $\mu < -\frac{1}{p}$  or  $\mu > 0$ , then  $\dot{B}_{p,\mu}(\mathbb{R}^n) = \Theta$ . Note that  $\dot{B}_{p,-\frac{1}{p}}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$  and  $\dot{B}_{p,0}(\mathbb{R}^n) = \dot{B}_p(\mathbb{R}^n)$ . Also define the weak central Morrey spaces  $W\dot{B}_{p,\mu}(\mathbb{R}^n) \equiv WLM_{p,n+n\mu}(\mathbb{R}^n)$ .

The following lemma, useful in itself, shows that the quasi-norm of the local Morrey spaces  $LL_{p,\lambda}^{\{x_0\}}$ ,  $\lambda \geq 0$  is equivalent to the quasi-norm  $\dot{B}_{p,\lambda}^{\{x_0\}}(\mathbb{R}^n)$ :

$$\|f\|_{\dot{B}_{p,\lambda}^{\{x_0\}}} = \sup_{k \in \mathbb{Z}} 2^{-\frac{k\lambda}{p}} \|f\chi_k\|_{L_p(\mathbb{R}^n)} < \infty,$$

where  $\chi_k$  is the characteristic function of the annulus  $B(x_0, 2^k) \setminus B(x_0, 2^{k-1})$ ,  $k \in \mathbb{Z}$ .

**Lemma 1.** *For  $0 < p \leq \infty$  and  $\lambda \geq 0$ , the quasi-norm  $\|f\|_{LL_{p,\lambda}^{\{x_0\}}}$  is equivalent to the quasi-norm  $\|f\|_{\dot{B}_{p,\lambda}^{\{x_0\}}}$ .*

*Proof.* Let  $0 < p \leq \infty$ ,  $\lambda \geq 0$  and  $f \in LL_{p,\lambda}^{\{x_0\}}(\mathbb{R}^n)$ . Then, it follows that

$$\|f\|_{\dot{B}_{p,\lambda}^{\{x_0\}}} \leq \sup_{k \in \mathbb{Z}} (2^k)^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x_0, 2^k))} \leq \sup_{r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x_0, r))} = \|f\|_{LL_{p,\lambda}^{\{x_0\}}}.$$

On the other hand, for  $0 < p < \infty$ , we get

$$\begin{aligned} \|f\|_{LL_{p,\lambda}^{\{x_0\}}}^p &= \sup_{k \in \mathbb{Z}} \sup_{2^{k-1} < r \leq 2^k} r^{-\lambda} \int_{B(x_0, r)} |f(y)|^p dy \\ &\leq 2^\lambda \sup_{k \in \mathbb{Z}} (2^k)^{-\lambda} \int_{B(x_0, 2^k)} |f(y)|^p dy \\ &= 2^\lambda \sup_{k \in \mathbb{Z}} 2^{-k\lambda} \sum_{d=-\infty}^k 2^{d\lambda} 2^{-d\lambda} \int_{B(x_0, 2^d) \setminus B(x_0, 2^{d-1})} |f(y)|^p dy \\ &\leq 2^\lambda \left( \sup_{d \in \mathbb{Z}} 2^{-d\lambda} \int_{B(x_0, 2^d) \setminus B(x_0, 2^{d-1})} |f(y)|^p dy \right) \left( \sup_{k \in \mathbb{Z}} 2^{-k\lambda} \sum_{d=-\infty}^k 2^{d\lambda} \right) \\ &= \frac{2^\lambda}{1 - 2^{-\lambda}} \|f\|_{\dot{B}_{p,\lambda}^{\{x_0\}}}^p. \end{aligned}$$

Thus, for  $0 < p < \infty$ , we have

$$\|f\|_{LL_{p,\lambda}^{\{x_0\}}} \leq 2^{\frac{\lambda}{p}} (1 - 2^{-\lambda})^{-\frac{1}{p}} \|f\|_{\dot{B}_{p,\lambda}^{\{x_0\}}}.$$

Also, for  $p = \infty$  we have

$$\|f\|_{LL_{\infty,\lambda}^{\{x_0\}}} \leq \|f\|_{\dot{B}_{\infty,\lambda}^{\{x_0\}}}.$$

□

In the case of  $\lambda = n$ , the quasi-norms  $\|f\|_{\dot{B}_{p,\lambda}^{\{x_0\}}}$  have been considered by Beurling [3] and Feichtinger [15].

Closely related to the above results, in this paper we prove the boundedness of the multi-sublinear operators  $T_\alpha^{(m)}$  ( $m \in \mathbb{N}$ ),  $\alpha \in (0, mn)$  satisfying condition (1.4) from product generalized local Morrey space  $LM_{p_1, \varphi_1}^{\{x_0\}} \times \cdots \times LM_{p_m, \varphi_m}^{\{x_0\}}$  to  $LM_{q, \varphi}^{\{x_0\}}$ , if  $1 < p_1, \dots, p_m < \infty$ ,  $\frac{1}{p} = \sum_{i=1}^m \frac{1}{p_i}$ ,  $\frac{1}{q_i} = \frac{1}{p_i} - \frac{\alpha}{mn}$  and  $\frac{1}{q} = \sum_{i=1}^m \frac{1}{q_i} = \frac{1}{p} - \frac{\alpha}{n}$ , and from the space  $LM_{p_1, \varphi_1}^{\{x_0\}} \times \cdots \times LM_{p_m, \varphi_m}^{\{x_0\}}$  to the weak space  $WLM_{q, \varphi}^{\{x_0\}}$ , if



$1 \leq p_1, \dots, p_m < \infty$ ,  $\frac{1}{p} = \sum_{i=1}^m \frac{1}{p_i}$ ,  $\frac{1}{q_i} = \frac{1}{p_i} - \frac{\alpha}{mn}$ ,  $\frac{1}{q} = \sum_{i=1}^m \frac{1}{q_i} = \frac{1}{p} - \frac{\alpha}{n}$  and at least one exponent  $p_i$  ( $i = 1, \dots, m$ ) equals 1. In the case of  $b_i \in LC_{q_i, \lambda_i}^{\{x_0\}}(\mathbb{R}^n)$  for  $0 \leq \lambda_i < \frac{1}{n}$ ,  $i = 1, \dots, m$ , we find the sufficient conditions on  $(\varphi_1, \dots, \varphi_m, \varphi)$  which ensures the boundedness of the commutator operators  $T_{\alpha, b}^{(m)}$  from  $LM_{p_1, \varphi_1}^{\{x_0\}} \times \dots \times LM_{p_m, \varphi_m}^{\{x_0\}}$  to  $LM_{q, \varphi}^{\{x_0\}}$ , where  $\frac{1}{q} = \sum_{i=1}^m \frac{1}{p_i} + \sum_{i=1}^m \frac{1}{q_i} - \frac{\alpha}{n}$ . In fact, in this paper the results of [24, 25, 26] (by taking  $\Omega \equiv 1$  there) will be generalized to the multilinear case; we omit the details here. But, the techniques and non-trivial estimates which have been used in the proofs of our main results are quite different from [24, 25]. For example, using inequality about the weighted Hardy operator  $H_w$  in [24, 25], in this paper we will only use the following relationship between essential supremum and essential infimum

$$(2.5) \quad \left( \operatorname{essinf}_{x \in E} f(x) \right)^{-1} = \operatorname{esssup}_{x \in E} \frac{1}{f(x)},$$

where  $f$  is any real-valued nonnegative function and measurable on  $E$  (see [59], page 143).

**Remark 1.** *Our results in this paper remain true for the inhomogeneous versions of local Campanato spaces  $LC_{q, \lambda}^{\{x_0\}}(\mathbb{R}^n)$  for  $0 \leq \lambda < \frac{1}{n}$  and generalized local Morrey spaces  $LM_{p, \varphi}^{\{x_0\}}$ .*

### 3. BOUNDEDNESS OF THE MULTI-SUBLINEAR OPERATORS $T_{\alpha}^{(m)}$ ( $m \in \mathbb{N}$ ) ON THE PRODUCT SPACES $LM_{p_1, \varphi_1}^{\{x_0\}} \times \dots \times LM_{p_m, \varphi_m}^{\{x_0\}}$

In this section we prove the boundedness of the operator  $T_{\alpha}^{(m)}$  ( $m \in \mathbb{N}$ ),  $\alpha \in (0, mn)$  satisfying condition (1.4) on the product generalized local Morrey spaces  $LM_{p_1, \varphi_1}^{\{x_0\}} \times \dots \times LM_{p_m, \varphi_m}^{\{x_0\}}$  by using (2.5) and the following Theorem 3.

We first prove the following Theorem 3.

**Theorem 3.** *Let  $x_0 \in \mathbb{R}^n$ ,  $0 < \alpha < mn$  and  $1 \leq p_i < \frac{mn}{\alpha}$  with  $\frac{1}{p} = \sum_{i=1}^m \frac{1}{p_i}$ ,*

*$\frac{1}{q_i} = \frac{1}{p_i} - \frac{\alpha}{mn}$  and  $\frac{1}{q} = \sum_{i=1}^m \frac{1}{q_i} = \frac{1}{p} - \frac{\alpha}{n}$ . Let  $T_{\alpha}^{(m)}$  ( $m \in \mathbb{N}$ ) be a multi-sublinear operator satisfying condition (1.4), bounded from  $L_{p_1} \times \dots \times L_{p_m}$  into  $L_q$  for  $p_i > 1$  ( $i = 1, \dots, m$ ), and bounded from  $L_{p_1} \times \dots \times L_{p_m}$  into the weak space  $WL_q$ , and at least one exponent  $p_i$  ( $i = 1, \dots, m$ ) equals 1.*

*Then, for  $p_i > 1$  ( $i = 1, \dots, m$ ) the inequality*

$$(3.1) \quad \left\| T_{\alpha}^{(m)}(\vec{f}) \right\|_{L_q(B_r)} \lesssim r^{\frac{n}{q}} \int_{2r}^{\infty} \prod_{i=1}^m \|f_i\|_{L_{p_i}(B(x_0, t))} t^{-\frac{n}{q}-1} dt$$

*holds for any ball  $B_r = B(x_0, r)$  and for all  $\vec{f} \in L_{p_1}^{loc}(\mathbb{R}^n) \times \dots \times L_{p_m}^{loc}(\mathbb{R}^n)$ .*

*Moreover, if at least one exponent  $p_i$  ( $i = 1, \dots, m$ ) equals 1, the inequality*

$$(3.2) \quad \left\| T_{\alpha}^{(m)}(\vec{f}) \right\|_{WL_q(B_r)} \lesssim r^{\frac{n}{q}} \int_{2r}^{\infty} \prod_{i=1}^m \|f_i\|_{L_{p_i}(B(x_0, t))} t^{-\frac{n}{q}-1} dt$$

holds for any ball  $B_r = B(x_0, r)$  and for all  $\vec{f} \in L_{p_1}^{loc}(\mathbb{R}^n) \times \cdots \times L_{p_m}^{loc}(\mathbb{R}^n)$ .

*Proof.* In order to simplify the proof, we consider only the situation when  $m = 2$ . Actually, a similar procedure works for all  $m \in \mathbb{N}$ . Thus, without loss of generality, it is sufficient to show that the conclusion holds for  $T_\alpha^{(2)}(\vec{f}) = T_\alpha^{(2)}(f_1, f_2)$ .

We just consider the case  $p_i > 1$  for  $i = 1, 2$ . For any  $x_0 \in \mathbb{R}^n$ , set  $B_r = B(x_0, r)$  for the ball centered at  $x_0$  and of radius  $r$  and  $B_{2r} = B(x_0, 2r)$ . Indeed, we also decompose  $f_i$  as  $f_i(y_i) = f_i(y_i)\chi_{B_{2r}} + f_i(y_i)\chi_{(B_{2r})^c}$  for  $i = 1, 2$ . And, we write  $f_1 = f_1^0 + f_1^\infty$  and  $f_2 = f_2^0 + f_2^\infty$ , where  $f_i^0 = f_i\chi_{B_{2r}}$ ,  $f_i^\infty = f_i\chi_{(B_{2r})^c}$ , for  $i = 1, 2$ . Thus, we have

$$\begin{aligned} \left\| T_\alpha^{(2)}(f_1, f_2) \right\|_{L_q(B_r)} &\leq \left\| T_\alpha^{(2)}(f_1^0, f_2^0) \right\|_{L_q(B_r)} + \left\| T_\alpha^{(2)}(f_1^0, f_2^\infty) \right\|_{L_q(B_r)} \\ &\quad + \left\| T_\alpha^{(2)}(f_1^\infty, f_2^0) \right\|_{L_q(B_r)} + \left\| T_\alpha^{(2)}(f_1^\infty, f_2^\infty) \right\|_{L_q(B_r)} \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Firstly, we use the boundedness of  $T_\alpha^{(2)}$  from  $L_{p_1} \times L_{p_2}$  into  $L_q$  to estimate  $I_1$ , and we obtain

$$\begin{aligned} I_1 &= \left\| T_\alpha^{(2)}(f_1^0, f_2^0) \right\|_{L_q(B_r)} \lesssim \|f_1\|_{L_{p_1}(B_{2r})} \|f_2\|_{L_{p_2}(B_{2r})} \\ &\lesssim r^{\frac{n}{q}} \|f_1\|_{L_{p_1}(B_{2r})} \|f_2\|_{L_{p_2}(B_{2r})} \int_{2r}^{\infty} \frac{dt}{t^{\frac{n}{q}+1}} \\ &\leq r^{\frac{n}{q}} \int_{2r}^{\infty} \prod_{i=1}^2 \|f_i\|_{L_{p_i}(B_t)} \frac{dt}{t^{\frac{n}{q}+1}}. \end{aligned}$$

Secondly, it is clear that  $|(x_0 - y_1, x_0 - y_2)|^{2n-\alpha} \geq |x_0 - y_2|^{2n-\alpha}$ . By the condition (1.4) with  $m = 2$ , Hölder's inequality, the estimate of  $I_2$  can be obtained as follows:

$$\begin{aligned} \left| T_\alpha^{(2)}(f_1^0, f_2^\infty)(x) \right| &\lesssim \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f_1^0(y_1)| |f_2^\infty(y_2)|}{|(x - y_1, x - y_2)|^{2n-\alpha}} dy_1 dy_2 \\ &\lesssim \int_{B_{2r}} |f_1(y_1)| dy_1 \int_{(B_{2r})^c} \frac{|f_2(y_2)|}{|x_0 - y_2|^{2n-\alpha}} dy_2 \\ &\approx \int_{B_{2r}} |f_1(y_1)| dy_1 \int_{(B_{2r})^c} |f_2(y_2)| \int_{|x_0 - y_2|}^{\infty} \frac{dt}{t^{2n-\alpha+1}} dy_2 \\ &\lesssim \|f_1\|_{L_{p_1}(B_{2r})} |B_{2r}|^{1-\frac{1}{p_1}} \int_{2r}^{\infty} \|f_2\|_{L_{p_2}(B_t)} |B_t|^{1-\frac{1}{p_2}} \frac{dt}{t^{2n-\alpha+1}} \\ &\lesssim \int_{2r}^{\infty} \prod_{i=1}^2 \|f_i\|_{L_{p_i}(B_t)} \frac{dt}{t^{\frac{n}{q}+1}}, \end{aligned}$$

where  $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ . Thus, the inequality

$$(3.3) \quad I_2 = \left\| T_\alpha^{(2)}(f_1^0, f_2^\infty) \right\|_{L_q(B_r)} \lesssim r^{\frac{n}{q}} \int_{2r}^{\infty} \prod_{i=1}^2 \|f_i\|_{L_{p_i}(B_t)} \frac{dt}{t^{\frac{n}{q}+1}}$$

is valid.

Similarly,  $I_3$  has the same estimate above, here we omit the details, thus the inequality

$$I_3 = \left\| T_\alpha^{(2)}(f_1^\infty, f_2^0) \right\|_{L_q(B_r)} \lesssim r^{\frac{n}{q}} \int_{2r}^{\infty} \prod_{i=1}^2 \|f_i\|_{L_{p_i}(B_t)} \frac{dt}{t^{\frac{n}{q}+1}}.$$

is valid.

At last, we consider the term  $I_4$ . Note that  $|(x_0 - y_1, x_0 - y_2)|^{2n-\alpha} \geq |x_0 - y_1|^{n-\frac{\alpha}{2}} |x_0 - y_2|^{n-\frac{\alpha}{2}}$ . Using the condition (1.4) with  $m = 2$  and by Hölder's inequality, we get

$$\begin{aligned} & \left| T_\alpha^{(2)}(f_1^\infty, f_2^\infty)(x) \right| \\ & \lesssim \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f_1(y_1) \chi_{(B_{2r})^c}| |f_2(y_2) \chi_{(B_{2r})^c}|}{|(x_0 - y_1, x_0 - y_2)|^{2n-\alpha}} dy_1 dy_2 \\ & \lesssim \int_{(B_{2r})^c} \int_{(B_{2r})^c} \frac{|f_1(y_1)| |f_2(y_2)|}{|x_0 - y_1|^{n-\frac{\alpha}{2}} |x_0 - y_2|^{n-\frac{\alpha}{2}}} dy_1 dy_2 \\ & \lesssim \sum_{j=1}^{\infty} \prod_{i=1}^2 \int_{B_{2^{j+1}r} \setminus B_{2^j r}} \frac{|f_i(y_i)|}{|x_0 - y_i|^{n-\frac{\alpha}{2}}} dy_i \\ & \lesssim \sum_{j=1}^{\infty} \prod_{i=1}^2 (2^j r)^{-n+\frac{\alpha}{2}} \int_{B_{2^{j+1}r}} |f_i(y_i)| dy_i \\ & \lesssim \sum_{j=1}^{\infty} (2^j r)^{-2n+\alpha} \prod_{i=1}^2 \|f_i\|_{L_{p_i}(B_{2^{j+1}r})} |B_{2^{j+1}r}|^{1-\frac{1}{p_i}} \\ & \lesssim \sum_{j=1}^{\infty} \int_{B_{2^{j+1}r}}^{2^{j+2}r} (2^{j+1}r)^{-2n+\alpha-1} \prod_{i=1}^2 \|f_i\|_{L_{p_i}(B_{2^{j+1}r})} |B_{2^{j+1}r}|^{1-\frac{1}{p_i}} dt \\ & \lesssim \sum_{j=1}^{\infty} \int_{B_{2^{j+1}r}}^{2^{j+2}r} \prod_{i=1}^2 \|f_i\|_{L_{p_i}(B_t)} |B_t|^{1-\frac{1}{p_i}} \frac{dt}{t^{2n+1-\alpha}} \\ & \lesssim \int_{2r}^{\infty} \prod_{i=1}^2 \|f_i\|_{L_{p_i}(B_t)} |B_t|^{2-(\frac{1}{p_1}+\frac{1}{p_2})} \frac{dt}{t^{2n+1-\alpha}} \\ & \lesssim \int_{2r}^{\infty} \prod_{i=1}^2 \|f_i\|_{L_{p_i}(B_t)} \frac{dt}{t^{\frac{n}{q}+1}}. \end{aligned}$$

Moreover, for  $p_1, p_2 \in [1, \infty)$  the inequality

$$(3.4) \quad I_4 = \left\| T_\alpha^{(2)}(f_1^\infty, f_2^\infty) \right\|_{L_q(B_r)} \lesssim r^{\frac{n}{q}} \int_{2r}^\infty \prod_{i=1}^2 \|f_i\|_{L_{p_i}(B_t)} \frac{dt}{t^{\frac{n}{q}+1}}$$

is valid.

By combining the above inequalities for  $I_1, I_2, I_3$  and  $I_4$  we obtain

$$\left\| T_\alpha^{(2)}(f_1, f_2) \right\|_{L_q(B_r)} \lesssim r^{\frac{n}{q}} \int_{2r}^\infty \prod_{i=1}^2 \|f_i\|_{L_{p_i}(B_t)} \frac{dt}{t^{\frac{n}{q}+1}}.$$

For the proof of the inequality (3.2), by a similar argument as in the proof of (3.1) and paying attention to the fact that  $\vec{f} \rightarrow T_\alpha^{(m)}(\vec{f})$  is bounded from  $L_{p_1} \times \cdots \times L_{p_m}$  to  $WL_q$ , we can similarly prove (3.2) so we omit the details here, which completes the proof.  $\square$

In the following theorem, which is one of our main results, we get the boundedness of the multi-sublinear operator  $T_\alpha^{(m)}$  ( $m \in \mathbb{N}$ ),  $\alpha \in (0, mn)$  satisfying condition (1.4) on the product generalized local Morrey spaces  $LM_{p_1, \varphi_1}^{\{x_0\}} \times \cdots \times LM_{p_m, \varphi_m}^{\{x_0\}}$ .

**Theorem 4.** (Our main result) Let  $x_0 \in \mathbb{R}^n$ ,  $0 < \alpha < mn$  and  $1 \leq p_i < \frac{mn}{\alpha}$  with  $\frac{1}{p} = \sum_{i=1}^m \frac{1}{p_i}$ ,  $\frac{1}{q_i} = \frac{1}{p_i} - \frac{\alpha}{mn}$  and  $\frac{1}{q} = \sum_{i=1}^m \frac{1}{q_i} = \frac{1}{p} - \frac{\alpha}{n}$ . Let  $T_\alpha^{(m)}$  ( $m \in \mathbb{N}$ ) be a multi-sublinear operator satisfying condition (1.4), bounded from  $L_{p_1} \times \cdots \times L_{p_m}$  into  $L_q$  for  $p_i > 1$  ( $i = 1, \dots, m$ ), and bounded from  $L_{p_1} \times \cdots \times L_{p_m}$  into the weak space  $WL_q$ , and at least one exponent  $p_i$  ( $i = 1, \dots, m$ ) equals 1. If functions  $\varphi, \varphi_i : \mathbb{R}^n \times (0, \infty) \rightarrow (0, \infty)$ , ( $i = 1, \dots, m$ ) and  $(\varphi_1, \dots, \varphi_m, \varphi)$  satisfy the condition

$$(3.5) \quad \int_r^\infty \frac{\operatorname{ess\,inf}_{t < \tau < \infty} \prod_{i=1}^m \varphi_i(x_0, \tau) \tau^{\frac{n}{p}}}{t^{\frac{n}{q}+1}} dt \leq C \varphi(x_0, r),$$

where  $C$  does not depend on  $r$ .

Then the operator  $T_\alpha^{(m)}$  is bounded from product space  $LM_{p_1, \varphi_1}^{\{x_0\}} \times \cdots \times LM_{p_m, \varphi_m}^{\{x_0\}}$  to  $LM_{q, \varphi}^{\{x_0\}}$  for  $p_i > 1$  ( $i = 1, \dots, m$ ) and from product  $LM_{p_1, \varphi_1}^{\{x_0\}} \times \cdots \times LM_{p_m, \varphi_m}^{\{x_0\}}$  to  $WLM_{q, \varphi}^{\{x_0\}}$  for  $p_i \geq 1$  ( $i = 1, \dots, m$ ). Moreover, we have for  $p_i > 1$  ( $i = 1, \dots, m$ )

$$(3.6) \quad \left\| T_\alpha^{(m)}(\vec{f}) \right\|_{LM_{q, \varphi}^{\{x_0\}}} \lesssim \prod_{i=1}^m \|f_i\|_{LM_{p_i, \varphi_i}^{\{x_0\}}},$$

and for  $p_i \geq 1$  ( $i = 1, \dots, m$ )

$$(3.7) \quad \left\| T_\alpha^{(m)}(\vec{f}) \right\|_{WLM_{q, \varphi}^{\{x_0\}}} \lesssim \prod_{i=1}^m \|f_i\|_{LM_{p_i, \varphi_i}^{\{x_0\}}}.$$

*Proof.* Since  $\vec{f} \in LM_{p_1, \varphi_1}^{\{x_0\}} \times \cdots \times LM_{p_m, \varphi_m}^{\{x_0\}}$ , by (2.5) and the non-decreasing, with respect to  $t$ , of the norm  $\prod_{i=1}^m \|f_i\|_{L_{p_i}(B_t)}$ , we get

$$\begin{aligned}
 & \frac{\prod_{i=1}^m \|f_i\|_{L_{p_i}(B_t)}}{\operatorname{essinf}_{0 < t < \tau < \infty} \prod_{i=1}^m \varphi_i(x_0, \tau) \tau^{\frac{n}{p}}} \\
 & \leq \operatorname{esssup}_{0 < t < \tau < \infty} \frac{\prod_{i=1}^m \|f_i\|_{L_{p_i}(B_t)}}{\prod_{i=1}^m \varphi_i(x_0, \tau) \tau^{\frac{n}{p}}} \\
 & \leq \operatorname{esssup}_{0 < \tau < \infty} \frac{\prod_{i=1}^m \|f_i\|_{L_{p_i}(B_\tau)}}{\prod_{i=1}^m \varphi_i(x_0, \tau) \tau^{\frac{n}{p}}} \\
 & \leq \prod_{i=1}^m \|f_i\|_{LM_{p_i, \varphi_i}^{\{x_0\}}}.
 \end{aligned} \tag{3.8}$$

For  $1 < p_1, \dots, p_m < \infty$ , since  $(\varphi_1, \dots, \varphi_m, \varphi)$  satisfies (3.5), we have

$$\begin{aligned}
 & \int_r^\infty \prod_{i=1}^m \|f_i\|_{L_{p_i}(B_t)} t^{-\frac{n}{q}-1} dt \\
 & \leq \int_r^\infty \frac{\prod_{i=1}^m \|f_i\|_{L_{p_i}(B_t)}}{\operatorname{essinf}_{t < \tau < \infty} \prod_{i=1}^m \varphi_i(x_0, \tau) \tau^{\frac{n}{p}}} \frac{\operatorname{essinf}_{t < \tau < \infty} \prod_{i=1}^m \varphi_i(x_0, \tau) \tau^{\frac{n}{p}}}{t^{\frac{n}{q}}} \frac{dt}{t} \\
 & \leq C \prod_{i=1}^m \|f_i\|_{LM_{p_i, \varphi_i}^{\{x_0\}}} \int_r^\infty \frac{\operatorname{essinf}_{t < \tau < \infty} \prod_{i=1}^m \varphi_i(x_0, \tau) \tau^{\frac{n}{p}}}{t^{\frac{n}{q}}} \frac{dt}{t} \\
 & \leq C \prod_{i=1}^m \|f_i\|_{LM_{p_i, \varphi_i}^{\{x_0\}}} \varphi(x_0, r).
 \end{aligned} \tag{3.9}$$

Then by (3.1) and (3.9), we get

$$\begin{aligned} \left\| T_\alpha^{(m)} \left( \vec{f} \right) \right\|_{LM_{q,\varphi}^{\{x_0\}}} &= \sup_{r>0} \varphi(x_0, r)^{-1} |B_r|^{-\frac{1}{q}} \left\| T_\alpha^{(m)} \left( \vec{f} \right) \right\|_{L_q(B_r)} \\ &\lesssim \sup_{r>0} \varphi(x_0, r)^{-1} \int_r^\infty \prod_{i=1}^m \|f_i\|_{L_{p_i}(B_t)} t^{-\frac{n}{q}} \frac{dt}{t} \\ &\lesssim \prod_{i=1}^m \|f_i\|_{LM_{p_i,\varphi_i}^{\{x_0\}}}. \end{aligned}$$

Thus we obtain (3.6). Also, for  $p_i = 1$  ( $i = 1, \dots, m$ ), the proof of the inequality (3.7) is similar and we omit the details here. Hence the proof is completed.  $\square$

Particularly, when  $\alpha = 0$ , we can get the following result for the  $m$ -linear Calderón-Zygmund operator by (1.2) and Theorem 1.

**Corollary 1.** *Let  $x_0 \in \mathbb{R}^n$ ,  $1 \leq p_1, \dots, p_m < \infty$  with  $\frac{1}{p} = \sum_{i=1}^m \frac{1}{p_i}$  and  $(\varphi_1, \dots, \varphi_m, \varphi)$  satisfies condition (3.5). Then the operators  $M^{(m)}$  and  $\overline{T}^{(m)}$  ( $m \in \mathbb{N}$ ) are bounded from product space  $LM_{p_1,\varphi_1}^{\{x_0\}} \times \dots \times LM_{p_m,\varphi_m}^{\{x_0\}}$  to  $LM_{p,\varphi}^{\{x_0\}}$  for  $p_i > 1$  ( $i = 1, \dots, m$ ) and from product  $LM_{p_1,\varphi_1}^{\{x_0\}} \times \dots \times LM_{p_m,\varphi_m}^{\{x_0\}}$  to  $WLM_{p,\varphi}^{\{x_0\}}$  for  $p_i \geq 1$  ( $i = 1, \dots, m$ ).*

**Remark 2.** *Note that, in the case of  $m = 1$  Theorem 4 and Corollary 1 have been proved in [4, 24, 25].*

If  $0 < \alpha < mn$  and  $\overline{T}_\alpha^{(m)}$  ( $m \in \mathbb{N}$ ) is an  $m$ -linear fractional integral operator, then the condition of (1.4) is obviously satisfied by (1.3). We can obtain the following corollary of Theorem 4 by Theorem 2:

**Corollary 2.** *Let  $x_0 \in \mathbb{R}^n$ ,  $0 < \alpha < mn$  and  $1 \leq p_i < \frac{mn}{\alpha}$  with  $\frac{1}{p} = \sum_{i=1}^m \frac{1}{p_i}$ ,*

*$\frac{1}{q_i} = \frac{1}{p_i} - \frac{\alpha}{mn}$  and  $\frac{1}{q} = \sum_{i=1}^m \frac{1}{q_i} = \frac{1}{p} - \frac{\alpha}{n}$  and also  $(\varphi_1, \dots, \varphi_m, \varphi)$  satisfies condition*

*(3.5). Then the operators  $M_\alpha^{(m)}$  and  $\overline{T}_\alpha^{(m)}$  ( $m \in \mathbb{N}$ ) are bounded from product space  $LM_{p_1,\varphi_1}^{\{x_0\}} \times \dots \times LM_{p_m,\varphi_m}^{\{x_0\}}$  to  $LM_{q,\varphi}^{\{x_0\}}$  for  $p_i > 1$  ( $i = 1, \dots, m$ ) and from product  $LM_{p_1,\varphi_1}^{\{x_0\}} \times \dots \times LM_{p_m,\varphi_m}^{\{x_0\}}$  to  $WLM_{q,\varphi}^{\{x_0\}}$  for  $p_i \geq 1$  ( $i = 1, \dots, m$ ).*

**Remark 3.** *Note that, in the case of  $m = 1$  Corollary 2 has been proved in [24, 25].*

#### 4. BOUNDEDNESS OF THE COMMUTATORS OF $m$ -LINEAR OPERATORS

GENERATED BY  $m$ -LINEAR FRACTIONAL INTEGRAL OPERATORS AND LOCAL CAMPANATO FUNCTIONS ON THE PRODUCT SPACES  $LM_{p_1,\varphi_1}^{\{x_0\}} \times \dots \times LM_{p_m,\varphi_m}^{\{x_0\}}$

In this section we prove the boundedness of the commutator operator  $T_{\alpha, \vec{b}}^{(m)}$  ( $m \in \mathbb{N}$ ) with  $\vec{b} \in LC_{q_i, \lambda_i}^{\{x_0\}}(\mathbb{R}^n)$  on the product generalized local Morrey spaces  $LM_{p_1,\varphi_1}^{\{x_0\}} \times \dots \times LM_{p_m,\varphi_m}^{\{x_0\}}$  by using (2.5) and the following Theorem 5.

Since  $BMO(\mathbb{R}^n) \subset \bigcap_{q>1} LC_q^{\{x_0\}}(\mathbb{R}^n)$ , if we only assume  $b \in LC_q^{\{x_0\}}(\mathbb{R}^n)$ , or more generally  $b \in LC_{q,\lambda}^{\{x_0\}}(\mathbb{R}^n)$ , then  $[b, \overline{T}]$  may not be a bounded operator on

$L_p(\mathbb{R}^n)$ ,  $1 < p < \infty$ . However, it has some boundedness properties on other spaces. As a matter of fact, Grafakos et al. [19] have considered the commutator with  $b \in LC_q^{\{x_0\}}(\mathbb{R}^n)$  on Herz spaces for the first time (see [16] for details). Moreover, in [4, 16, 24, 25, 26, 54] they have considered the commutators with  $b \in LC_{q,\lambda}^{\{x_0\}}(\mathbb{R}^n)$ .

There are two major reasons for considering the problem of commutators. The first one is that the boundedness of commutators can produce some characterizations of function spaces (see [4, 6, 24, 25, 26, 27, 29, 44, 49]). The other one is that the theory of commutators plays an important role in the study of the regularity of solutions to elliptic and parabolic PDEs of the second order (see [9, 10, 43, 51]). The boundedness of the commutator has also been generalized to other contexts and important applications to some non-linear PDEs have been given by Coifman et al. [13].

The definition of local Campanato space is as follows.

**Definition 3.** [4, 24, 27] Let  $1 \leq q < \infty$  and  $0 \leq \lambda < \frac{1}{n}$ . A function  $f \in L_q^{loc}(\mathbb{R}^n)$  is said to belong to the  $LC_{q,\lambda}^{\{x_0\}}(\mathbb{R}^n)$  (local Campanato space), if

$$(4.1) \quad \|f\|_{LC_{q,\lambda}^{\{x_0\}}} = \sup_{r>0} \left( \frac{1}{|B(x_0, r)|^{1+\lambda q}} \int_{B(x_0, r)} |f(y) - f_{B(x_0, r)}|^q dy \right)^{\frac{1}{q}} < \infty,$$

where

$$f_{B(x_0, r)} = \frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} f(y) dy.$$

Define

$$LC_{q,\lambda}^{\{x_0\}}(\mathbb{R}^n) = \left\{ f \in L_q^{loc}(\mathbb{R}^n) : \|f\|_{LC_{q,\lambda}^{\{x_0\}}} < \infty \right\}.$$

**Remark 4.** If two functions which differ by a constant are regarded as a function in the space  $LC_{q,\lambda}^{\{x_0\}}(\mathbb{R}^n)$ , then  $LC_{q,\lambda}^{\{x_0\}}(\mathbb{R}^n)$  becomes a Banach space. The space  $LC_{q,\lambda}^{\{x_0\}}(\mathbb{R}^n)$  when  $\lambda = 0$  is just the  $LC_q^{\{x_0\}}(\mathbb{R}^n)$ . Apparently, (4.1) is equivalent to the following condition:

$$\sup_{r>0} \inf_{c \in \mathbb{C}} \left( \frac{1}{|B(x_0, r)|^{1+\lambda q}} \int_{B(x_0, r)} |f(y) - c|^q dy \right)^{\frac{1}{q}} < \infty.$$

In [34], Lu and Yang have introduced the central BMO space  $CBMO_q(\mathbb{R}^n) = LC_{q,0}^{\{0\}}(\mathbb{R}^n)$ . Also the space  $CBMO^{\{x_0\}}(\mathbb{R}^n) = LC_{1,0}^{\{x_0\}}(\mathbb{R}^n)$  has been considered in other denotes in [47]. The space  $LC_q^{\{x_0\}}(\mathbb{R}^n)$  can be regarded as a local version of  $BMO(\mathbb{R}^n)$ , the space of bounded mean oscillation, at the origin. But, they have quite different properties. The classical John-Nirenberg inequality shows that functions in  $BMO(\mathbb{R}^n)$  are locally exponentially integrable. This implies that, for any  $1 \leq q < \infty$ , the functions in  $BMO(\mathbb{R}^n)$  can be described by means of the condition:

$$\sup_{B \subset \mathbb{R}^n} \left( \frac{1}{|B|} \int_B |f(y) - f_B|^q dy \right)^{1/q} < \infty,$$

where  $B$  denotes an arbitrary ball on  $\mathbb{R}^n$ . However, the space  $LC_q^{\{x_0\}}(\mathbb{R}^n)$  depends on  $q$ . If  $q_1 < q_2$ , then  $LC_{q_2}^{\{x_0\}}(\mathbb{R}^n) \subsetneq LC_{q_1}^{\{x_0\}}(\mathbb{R}^n)$ . Therefore, there is no analogy of the famous John-Nirenberg inequality of  $BMO(\mathbb{R}^n)$  for the space  $LC_q^{\{x_0\}}(\mathbb{R}^n)$ . One can imagine that the behavior of  $LC_q^{\{x_0\}}(\mathbb{R}^n)$  may be quite different from that of  $BMO(\mathbb{R}^n)$  (see [37] for details).

**Lemma 2.** [4, 24, 25] *Let  $b$  be function in  $LC_{q,\lambda}^{\{x_0\}}(\mathbb{R}^n)$ ,  $1 \leq q < \infty$ ,  $0 \leq \lambda < \frac{1}{n}$  and  $r_1, r_2 > 0$ . Then*

$$(4.2) \quad \left( \frac{1}{|B(x_0, r_1)|^{1+\lambda q}} \int_{B(x_0, r_1)} |b(y) - b_{B(x_0, r_2)}|^q dy \right)^{\frac{1}{q}} \leq C \left( 1 + \left| \ln \frac{r_1}{r_2} \right| \right) \|b\|_{LC_{q,\lambda}^{\{x_0\}}},$$

where  $C > 0$  is independent of  $b, r_1$  and  $r_2$ .

From this inequality (4.2), we have for  $0 < r_2 < r_1$

$$(4.3) \quad |b_{B(x_0, r_1)} - b_{B(x_0, r_2)}| \leq C \left( 1 + \ln \frac{r_1}{r_2} \right) |B(x_0, r_1)|^\lambda \|b\|_{LC_{q,\lambda}^{\{x_0\}}}$$

and it is easy to see that

$$(4.4) \quad \|b - (b)_{B(x_0, r)}\|_{L_q(B(x_0, r))} \leq C r^{\frac{n}{q} + n\lambda} \|b\|_{LC_{q,\lambda}^{\{x_0\}}}.$$

Note that one gets the proof of (4.2) in a similar way as in Lemma 2.1 of [33].

In [14] the boundedness of commutator operators satisfying condition (1.4) (by taking  $m = 1$  there) has been proved.

About the commutator of multilinear operators  $T_\alpha^{(m)}$  satisfying condition (1.4), we get the following corresponding theorem.

**Theorem 5.** *Let  $x_0 \in \mathbb{R}^n$ ,  $0 < \alpha < mn$ , and  $1 \leq p_i < \frac{mn}{\alpha}$  with  $\frac{1}{q} = \sum_{i=1}^m \frac{1}{p_i} + \sum_{i=1}^m \frac{1}{q_i} - \frac{\alpha}{n}$  and  $\vec{b} \in LC_{q_i, \lambda_i}^{\{x_0\}}(\mathbb{R}^n)$  for  $0 \leq \lambda_i < \frac{1}{n}$ ,  $i = 1, \dots, m$ . Let also,  $T_\alpha^{(m)}$  ( $m \in \mathbb{N}$ ) be a multilinear operator satisfying condition (1.4), bounded from  $L_{p_1} \times \dots \times L_{p_m}$  into  $L_q$ . Then the inequality*

$$(4.5) \quad \|T_{\alpha, \vec{b}}^{(m)}(\vec{f})\|_{L_q(B_r)} \lesssim \prod_{i=1}^m \|\vec{b}\|_{LC_{q_i, \lambda_i}^{\{x_0\}}} r^{\frac{n}{q}} \times \int_{2r}^{\infty} \left( 1 + \ln \frac{t}{r} \right)^m t^{n \left( -\frac{1}{q} + \sum_{i=1}^m \lambda_i + \sum_{i=1}^m \frac{1}{q_i} \right) - 1} \prod_{i=1}^m \|f_i\|_{L_{p_i}(B_t)} dt$$

holds for any ball  $B_r = B(x_0, r)$  and for all  $\vec{f} \in L_{p_1}^{loc}(\mathbb{R}^n) \times \dots \times L_{p_m}^{loc}(\mathbb{R}^n)$ .

*Proof.* As in the proof of Theorem 3, we consider only the situation when  $m = 2$ . Actually, a similar procedure works for all  $m \in \mathbb{N}$ . Thus, without loss of generality, it is sufficient to show that the conclusion holds for  $T_{\alpha, \vec{b}}^{(2)}(\vec{f}) = T_{\alpha, (b_1, b_2)}^{(2)}(f_1, f_2)$ .

We just consider the case  $p_i, q_i > 1$  for  $i = 1, 2$ . For any  $x_0 \in \mathbb{R}^n$ , set  $B_r = B(x_0, r)$  for the ball centered at  $x_0$  and of radius  $r$  and  $B_{2r} = B(x_0, 2r)$ . Thus, we



have the following decomposition,

$$\begin{aligned}
T_{\alpha, (b_1, b_2)}^{(2)}(f_1, f_2)(x) &= (b_1 - (b_1)_{B_r})(b_2 - (b_2)_{B_r}) T_{\alpha}^{(2)}(f_1, f_2)(x) \\
&\quad - (b_1 - (b_1)_{B_r}) T_{\alpha}^{(2)}[f_1, (b_2(\cdot) - (b_2)_{B_r}) f_2](x) \\
&\quad - (b_2 - (b_2)_{B_r}) T_{\alpha}^{(2)}[(b_1(\cdot) - (b_1)_{B_r}) f_1, f_2](x) \\
&\quad + T_{\alpha}^{(2)}[(b_1(\cdot) - (b_1)_{B_r}) f_1, (b_2(\cdot) - (b_2)_{B_r}) f_2](x) \\
&\equiv H_1(x) + H_2(x) + H_3(x) + H_4(x).
\end{aligned}$$

Thus,

$$\begin{aligned}
\|T_{\alpha, (b_1, b_2)}^{(2)}(f_1, f_2)\|_{L_q(B_r)} &= \left( \int_{B_r} |T_{\alpha, (b_1, b_2)}^{(2)}(f_1, f_2)(x)|^q dx \right)^{\frac{1}{q}} \\
(4.6) \qquad \qquad \qquad &\leq \sum_{i=1}^4 \left( \int_{B_r} |H_i(x)|^q dx \right)^{\frac{1}{q}} = \sum_{i=1}^4 G_i.
\end{aligned}$$

One observes that the estimate of  $G_2$  is analogous to that of  $G_3$ . Thus, we will only estimate  $G_1$ ,  $G_2$  and  $G_4$ .

Indeed, we also decompose  $f_i$  as  $f_i(y_i) = f_i(y_i) \chi_{B_{2r}} + f_i(y_i) \chi_{(B_{2r})^c}$  for  $i = 1, 2$ . And, we write  $f_1 = f_1^0 + f_1^\infty$  and  $f_2 = f_2^0 + f_2^\infty$ , where  $f_i^0 = f_i \chi_{B_{2r}}$ ,  $f_i^\infty = f_i \chi_{(B_{2r})^c}$ , for  $i = 1, 2$ .

(i) For  $G_1 = \| (b_1 - (b_1)_{B_r})(b_2 - (b_2)_{B_r}) T_{\alpha, (b_1, b_2)}^{(2)}(f_1^0, f_2^0) \|_{L_q(B_r)}$ , we decompose it into four parts as follows:

$$\begin{aligned}
G_1 &\lesssim \left\| (b_1 - (b_1)_{B_r})(b_2 - (b_2)_{B_r}) T_{\alpha}^{(2)}(f_1^0, f_2^0) \right\|_{L_q(B_r)} \\
&\quad + \left\| (b_1 - (b_1)_{B_r}) T_{\alpha}^{(2)}[f_1^0, (b_2 - (b_2)_{B_r}) f_2^0] \right\|_{L_q(B_r)} \\
&\quad + \left\| (b_2 - (b_2)_{B_r}) T_{\alpha}^{(2)}[(b_1 - (b_1)_{B_r}) f_1^0, f_2^0] \right\|_{L_q(B_r)} \\
&\quad + \left\| T_{\alpha}^{(2)}[(b_1 - (b_1)_{B_r}) f_1^0, (b_2 - (b_2)_{B_r}) f_2^0] \right\|_{L_q(B_r)} \\
&\equiv G_{11} + G_{12} + G_{13} + G_{14}.
\end{aligned}$$

Firstly,  $1 < \overline{p}, \overline{q} < \infty$ , such that  $\frac{1}{\overline{p}} = \frac{1}{p_1} + \frac{1}{p_2}$  and  $\frac{1}{\overline{q}} = \frac{1}{\overline{p}} - \frac{\alpha}{n}$ ,  $\frac{1}{q} = \frac{1}{\overline{p}} + \frac{1}{\overline{q}}$ ,  $\frac{1}{\overline{r}} = \frac{1}{q_1} + \frac{1}{q_2}$ . Then, using Hölder's inequality and from the boundedness of  $T_{\alpha}^{(2)}$  from  $L_{p_1} \times L_{p_2}$

into  $L_{\overline{q}}$  it follows that:

$$\begin{aligned}
G_{11} &\lesssim \|(b_1 - (b_1)_{B_r})(b_2 - (b_2)_{B_r})\|_{L_{\overline{q}}(B_r)} \left\| T_{\alpha}^{(2)}(f_1^0, f_2^0) \right\|_{L_{\overline{q}}(B_r)} \\
&\lesssim \|b_1 - (b_1)_B\|_{L_{q_1}(B_r)} \|b_2 - (b_2)_B\|_{L_{q_2}(B_r)} \|f_1\|_{L_{p_1}(B_{2r})} \|f_2\|_{L_{p_2}(B_{2r})} \\
&\lesssim \|b_1 - (b_1)_{B_r}\|_{L_{q_1}(B_r)} \|b_2 - (b_2)_{B_r}\|_{L_{q_2}(B_r)} r^{n\left(\frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha}{n}\right)} \\
&\times \int_{2r}^{\infty} \prod_{i=1}^2 \|f_i\|_{L_{p_i}(B_t)} \frac{dt}{t^{n\left(\frac{1}{p_1} + \frac{1}{p_2}\right) + 1 - \alpha}} \\
&\lesssim \|b_1\|_{LC_{q_1, \lambda_1}^{\{x_0\}}} \|b_2\|_{LC_{q_2, \lambda_2}^{\{x_0\}}} r^{n\left(\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha}{n}\right)} \\
&\times \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 t^{n(\lambda_1 + \lambda_2) - n\left(\frac{1}{p_1} + \frac{1}{p_2}\right) - 1 + \alpha} \prod_{i=1}^2 \|f_i\|_{L_{p_i}(B_t)} dt \\
&\lesssim \|b_1\|_{LC_{q_1, \lambda_1}^{\{x_0\}}} \|b_2\|_{LC_{q_2, \lambda_2}^{\{x_0\}}} r^{\frac{n}{q}} \\
&\times \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 t^{n(\lambda_1 + \lambda_2) - \frac{n}{q} + n\left(\frac{1}{q_1} + \frac{1}{q_2}\right) - 1} \prod_{i=1}^2 \|f_i\|_{L_{p_i}(B_t)} dt.
\end{aligned}$$

Secondly, for  $G_{12}$ , let  $1 < \tau < \infty$ , such that  $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{\tau}$ . Then similar to the estimates for  $G_{11}$ , we have

$$\begin{aligned}
G_{12} &\lesssim \|b_1 - (b_1)_{B_r}\|_{L_{q_1}(B_r)} \left\| T_{\alpha}^{(2)}[f_1^0, (b_2 - (b_2)_{B_r}) f_2^0] \right\|_{L_{\tau}(B_r)} \\
&\lesssim \|b_1 - (b_1)_{B_r}\|_{L_{q_1}(B_r)} \|f_1^0\|_{L_{p_1}(\mathbb{R}^n)} \|(b_2 - (b_2)_{B_r}) f_2^0\|_{L_k(\mathbb{R}^n)} \\
&\lesssim \|b_1 - (b_1)_{B_r}\|_{L_{q_1}(B_r)} \|b_2 - (b_2)_{B_r}\|_{L_{q_2}(B_{2r})} \|f_1\|_{L_{p_1}(B_{2r})} \|f_2\|_{L_{p_2}(B_{2r})},
\end{aligned}$$

where  $1 < k < \frac{2n}{\alpha}$ , such that  $\frac{1}{k} = \frac{1}{p_2} + \frac{1}{q_2} = \frac{1}{\tau} - \frac{1}{p_1} + \frac{\alpha}{n}$ . From Lemma 2, it is easy to see that

$$\|b_i - (b_i)_{B_r}\|_{L_{q_i}(B_r)} \leq C r^{\frac{n}{q_i} + n\lambda_i} \|b_i\|_{LC_{q_i, \lambda_i}^{\{x_0\}}},$$

and

$$\begin{aligned}
\|b_i - (b_i)_{B_r}\|_{L_{q_i}(B_{2r})} &\leq \|b_i - (b_i)_{B_{2r}}\|_{L_{q_i}(B_{2r})} + \|(b_i)_{B_r} - (b_i)_{B_{2r}}\|_{L_{q_i}(B_{2r})} \\
(4.7) \quad &\lesssim r^{\frac{n}{q_i} + n\lambda_i} \|b_i\|_{LC_{q_i, \lambda_i}^{\{x_0\}}},
\end{aligned}$$

for  $i = 1, 2$ . Hence, we get

$$\begin{aligned}
G_{12} &\lesssim \|b_1\|_{LC_{q_1, \lambda_1}^{\{x_0\}}} \|b_2\|_{LC_{q_2, \lambda_2}^{\{x_0\}}} r^{\frac{n}{q}} \\
&\times \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 t^{n(\lambda_1 + \lambda_2) - \frac{n}{q} + n\left(\frac{1}{q_1} + \frac{1}{q_2}\right) - 1} \prod_{i=1}^2 \|f_i\|_{L_{p_i}(B_t)} dt.
\end{aligned}$$

Similarly,  $G_{13}$  has the same estimate as above, here we omit the details, thus the inequality

$$\begin{aligned} G_{13} &\lesssim \|b_1\|_{LC_{q_1, \lambda_1}^{\{x_0\}}} \|b_2\|_{LC_{q_2, \lambda_2}^{\{x_0\}}} r^{\frac{n}{q}} \\ &\quad \times \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 t^{n(\lambda_1 + \lambda_2) - \frac{n}{q} + n\left(\frac{1}{q_1} + \frac{1}{q_2}\right) - 1} \prod_{i=1}^2 \|f_i\|_{L_{p_i}(B_t)} dt \end{aligned}$$

is valid.

At last, we consider the term  $G_{14}$ . Let  $1 < \tau_1, \tau_2 < \frac{2n}{\alpha}$ , such that  $\frac{1}{\tau_1} = \frac{1}{p_1} + \frac{1}{q_1}$ ,  $\frac{1}{\tau_2} = \frac{1}{p_2} + \frac{1}{q_2}$  and  $\frac{1}{q} = \frac{1}{\tau_1} + \frac{1}{\tau_2} - \frac{\alpha}{n}$ . Then by the boundedness of  $T_{\alpha}^{(2)}$  from  $L_{\tau_1} \times L_{\tau_2}$  into  $L_q$ , Hölder's inequality and (4.7), we obtain

$$\begin{aligned} G_{14} &\lesssim \|(b_1 - (b_1)_{B_r}) f_1^0\|_{L_{\tau_1}(B_r)} \|(b_2 - (b_2)_{B_r}) f_2^0\|_{L_{\tau_2}(B_r)} \\ &\lesssim \|b_1 - (b_1)_{B_r}\|_{L_{q_1}(B_{2r})} \|b_2 - (b_2)_{B_r}\|_{L_{q_2}(B_{2r})} \|f_1\|_{L_{p_1}(B_{2r})} \|f_2\|_{L_{p_2}(B_{2r})} \\ &\lesssim \|b_1\|_{LC_{q_1, \lambda_1}^{\{x_0\}}} \|b_2\|_{LC_{q_2, \lambda_2}^{\{x_0\}}} r^{\frac{n}{q}} \\ &\quad \times \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 t^{n(\lambda_1 + \lambda_2) - \frac{n}{q} + n\left(\frac{1}{q_1} + \frac{1}{q_2}\right) - 1} \prod_{i=1}^2 \|f_i\|_{L_{p_i}(B_t)} dt. \end{aligned}$$

Combining all the estimates of  $G_{11}$ ,  $G_{12}$ ,  $G_{13}$ ,  $G_{14}$ ; we get

$$\begin{aligned} G_1 &= \left\| T_{\alpha, (b_1, b_2)}^{(2)} (f_1^0, f_2^0) \right\|_{L_q(B_r)} \lesssim \|b_1\|_{LC_{q_1, \lambda_1}^{\{x_0\}}} \|b_2\|_{LC_{q_2, \lambda_2}^{\{x_0\}}} r^{\frac{n}{q}} \\ &\quad \times \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 t^{n(\lambda_1 + \lambda_2) - \frac{n}{q} + n\left(\frac{1}{q_1} + \frac{1}{q_2}\right) - 1} \prod_{i=1}^2 \|f_i\|_{L_{p_i}(B_t)} dt. \end{aligned}$$

(ii) For  $G_2 = \left\| T_{\alpha, (b_1, b_2)}^{(2)} (f_1^0, f_2^{\infty}) \right\|_{L_q(B_r)}$ , we also write

$$\begin{aligned} G_2 &\lesssim \left\| (b_1 - (b_1)_{B_r}) (b_2 - (b_2)_{B_r}) T_{\alpha}^{(2)} (f_1^0, f_2^{\infty}) \right\|_{L_q(B_r)} \\ &\quad + \left\| (b_1 - (b_1)_{B_r}) T_{\alpha}^{(2)} [f_1^0, (b_2 - (b_2)_{B_r}) f_2^{\infty}] \right\|_{L_q(B_r)} \\ &\quad + \left\| (b_2 - (b_2)_{B_r}) T_{\alpha}^{(2)} [(b_1 - (b_1)_{B_r}) f_1^0, f_2^{\infty}] \right\|_{L_q(B_r)} \\ &\quad + \left\| T_{\alpha}^{(2)} [(b_1 - (b_1)_{B_r}) f_1^0, (b_2 - (b_2)_{B_r}) f_2^{\infty}] \right\|_{L_q(B_r)} \\ &\equiv G_{21} + G_{22} + G_{23} + G_{24}. \end{aligned}$$

Let  $1 < p_1, p_2 < \frac{2n}{\alpha}$ , such that  $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$  and  $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ . Then, using Hölder's inequality and noting that in (3.3)  $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha}{n}$ , we have

$$\begin{aligned}
G_{21} &= \left\| (b_1 - (b_1)_{B_r}) (b_2 - (b_2)_{B_r}) T_\alpha^{(2)} (f_1^0, f_2^\infty) \right\|_{L_q(B_r)} \\
&\lesssim \left\| (b_1 - (b_1)_{B_r}) (b_2 - (b_2)_{B_r}) \right\|_{L_{\overline{r}}(B_r)} \left\| T_\alpha^{(2)} (f_1^0, f_2^\infty) \right\|_{L_{\overline{q}}(B_r)} \\
&\lesssim \|b_1 - (b_1)_{B_r}\|_{L_{q_1}(B_r)} \|b_2 - (b_2)_{B_r}\|_{L_{q_2}(B_r)} \\
&\quad \times r^{\frac{n}{q}} \int_{2r}^{\infty} \|f_1\|_{L_{p_1}(B_t)} \|f_2\|_{L_{p_2}(B_t)} t^{-n(\frac{1}{p_1} + \frac{1}{p_2}) + \alpha - 1} dt \\
&\lesssim \|b_1\|_{LC_{q_1, \lambda_1}^{\{x_0\}}} \|b_2\|_{LC_{q_2, \lambda_2}^{\{x_0\}}} r^{n(\frac{1}{q_1} + \frac{1}{q_2}) + n(\lambda_1 + \lambda_2)} r^{n(\frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha}{n})} \\
&\quad \times \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 t^{-n(\frac{1}{p_1} + \frac{1}{p_2}) + \alpha - 1} \prod_{i=1}^2 \|f_i\|_{L_{p_i}(B_t)} dt \\
&\lesssim \|b_1\|_{LC_{q_1, \lambda_1}^{\{x_0\}}} \|b_2\|_{LC_{q_2, \lambda_2}^{\{x_0\}}} r^{\frac{n}{q}} \\
&\quad \times \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 t^{n(\lambda_1 + \lambda_2) - \frac{n}{q} + n(\frac{1}{q_1} + \frac{1}{q_2}) - 1} \prod_{i=1}^2 \|f_i\|_{L_{p_i}(B_t)} dt
\end{aligned}$$

where  $\frac{1}{q} = \frac{1}{p} + \frac{1}{q}$ ,  $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ .

For estimate  $G_{22}$ , we use condition (1.4) with  $m = 2$  and we get

$$\begin{aligned}
&\left| T_\alpha^{(2)} [f_1^0, (b_2(\cdot) - (b_2)_{B_r}) f_2^\infty] (x) \right| \\
&\lesssim \int_{B_{2r}} |f_1(y_1)| dy_1 \int_{(B_{2r})^c} \frac{|b_2(y_2) - (b_2)_B| |f_2(y_2)|}{|x_0 - y_2|^{2n - \alpha}} dy_2.
\end{aligned}$$

It's obvious that

$$(4.8) \quad \int_{B_{2r}} |f_1(y_1)| dy_1 \lesssim \|f_1\|_{L_{p_1}(B_{2r})} |B_{2r}|^{1 - \frac{1}{p_1}},$$

and using Hölder's inequality and by (4.3) and (4.7) we have

$$\begin{aligned}
& \int_{(B_{2r})^c} \frac{|b_2(y_2) - (b_2)_B| |f_2(y_2)|}{|x_0 - y_2|^{2n-\alpha}} dy_2 \\
& \lesssim \int_{(B_{2r})^c} |b_2(y_2) - (b_2)_{B_r}| |f_2(y_2)| \left[ \int_{|x_0 - y_2|}^{\infty} \frac{dt}{t^{2n-\alpha+1}} \right] dy_2 \\
& \lesssim \int_{2r}^{\infty} \|b_2(y_2) - (b_2)_{B_t}\|_{L_{q_2}(B_t)} \|f_2\|_{L_{p_2}(B_t)} |B_t|^{1 - (\frac{1}{p_2} + \frac{1}{q_2})} \frac{dt}{t^{2n-\alpha+1}} \\
& \quad + \int_{2r}^{\infty} |(b_2)_{B_t} - (b_2)_{B_r}| \|f_2\|_{L_{p_2}(B_t)} |B_t|^{1 - \frac{1}{p_2}} \frac{dt}{t^{2n-\alpha+1}} \\
& \lesssim \|b_2\|_{LC_{q_2, \lambda_2}^{\{x_0\}}} \int_{2r}^{\infty} |B_t|^{\frac{1}{q_2} + \lambda_2} \|f_2\|_{L_{p_2}(B_t)} |B_t|^{1 - (\frac{1}{p_2} + \frac{1}{q_2})} \frac{dt}{t^{2n-\alpha+1}} \\
& \quad + \|b_2\|_{LC_{q_2, \lambda_2}^{\{x_0\}}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) |B_t|^{\lambda_2} \|f_2\|_{L_{p_2}(B_t)} |B_t|^{1 - \frac{1}{p_2}} \frac{dt}{t^{2n-\alpha+1}} \\
(4.9) \quad & \lesssim \|b_2\|_{LC_{q_2, \lambda_2}^{\{x_0\}}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 t^{-n+n\lambda_2 - \frac{n}{p_2} - 1 + \alpha} \|f_2\|_{L_{p_2}(B_t)} dt.
\end{aligned}$$

Hence, by (4.8) and (4.9), it follows that:

$$\begin{aligned}
& \left| T_{\alpha}^{(2)} [f_1^0, (b_2(\cdot) - (b_2)_{B_r}) f_2^{\infty}] (x) \right| \\
& \lesssim \|b_2\|_{LC_{q_2, \lambda_2}^{\{x_0\}}} \|f_1\|_{L_{p_1}(B_{2r})} |B_{2r}|^{1 - \frac{1}{p_1}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 t^{-n+n\lambda_2 - \frac{n}{p_2} - 1 + \alpha} \|f_2\|_{L_{p_2}(B_t)} dt \\
& \lesssim \|b_2\|_{LC_{q_2, \lambda_2}^{\{x_0\}}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 t^{n\lambda_2 - n(\frac{1}{p_1} + \frac{1}{p_2}) - 1 + \alpha} \prod_{i=1}^2 \|f_i\|_{L_{p_i}(B_t)} dt.
\end{aligned}$$

Thus, let  $1 < \tau < \infty$ , such that  $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{\tau}$ . Then similar to the estimates for  $G_{11}$ , we have

$$\begin{aligned}
G_{22} &= \left\| (b_1 - (b_1)_{B_r}) T_\alpha^{(2)} [f_1^0, (b_2 - (b_2)_{B_r}) f_2^\infty] \right\|_{L_q(B_r)} \\
&\lesssim \|b_1 - (b_1)_{B_r}\|_{L_{q_1}(B_r)} \left\| T_\alpha^{(2)} [f_1^0, (b_2 - (b_2)_{B_r}) f_2^\infty] \right\|_{L_\tau(B_r)} \\
&\lesssim \|b_1\|_{LC_{q_1, \lambda_1}^{\{x_0\}}} \|b_2\|_{LC_{q_2, \lambda_2}^{\{x_0\}}} |B_r|^{\lambda_1 + \frac{1}{q_1} + \frac{1}{\tau}} \\
&\quad \times \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 t^{n\lambda_2 - n(\frac{1}{p_1} + \frac{1}{p_2}) - 1 + \alpha} \prod_{i=1}^2 \|f_i\|_{L_{p_i}(B_t)} dt \\
&\lesssim \|b_1\|_{LC_{q_1, \lambda_1}^{\{x_0\}}} \|b_2\|_{LC_{q_2, \lambda_2}^{\{x_0\}}} r^{\frac{n}{q}} \\
&\quad \times \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 t^{n(\lambda_1 + \lambda_2) - \frac{n}{q} + n(\frac{1}{q_1} + \frac{1}{q_2}) - 1} \prod_{i=1}^2 \|f_i\|_{L_{p_i}(B_t)} dt.
\end{aligned}$$

Similarly,  $G_{23}$  has the same estimate above, here we omit the details, thus the inequality

$$\begin{aligned}
G_{23} &= \left\| (b_2 - (b_2)_{B_r}) T_\alpha^{(2)} [(b_1 - (b_1)_{B_r}) f_1^0, f_2^\infty] \right\|_{L_q(B_r)} \\
&\lesssim \|b_1\|_{LC_{q_1, \lambda_1}^{\{x_0\}}} \|b_2\|_{LC_{q_2, \lambda_2}^{\{x_0\}}} r^{\frac{n}{q}} \\
&\quad \times \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 t^{n(\lambda_1 + \lambda_2) - \frac{n}{q} + n(\frac{1}{q_1} + \frac{1}{q_2}) - 1} \prod_{i=1}^2 \|f_i\|_{L_{p_i}(B_t)} dt
\end{aligned}$$

is valid.

Now, using the condition (1.4) with  $m = 2$ , we have

$$\begin{aligned}
&\left| T_\alpha^{(2)} [(b_1 - (b_1)_{B_r}) f_1^0, (b_2 - (b_2)_{B_r}) f_2^\infty] (x) \right| \\
&\lesssim \int_{B_{2r}} |b_1(y_1) - (b_1)_{B_r}| |f_1(y_1)| dy_1 \int_{(B_{2r})^c} \frac{|b_2(y_2) - (b_2)_{B_r}| |f_2(y_2)|}{|x_0 - y_2|^{2n-\alpha}} dy_2.
\end{aligned}$$

It's obvious that from Hölder's inequality and (4.4)

$$(4.10) \quad \int_{B_{2r}} |b_1(y_1) - (b_1)_{B_r}| |f_1(y_1)| dy_1 \lesssim \|b_1\|_{LC_{q_1, \lambda_1}^{\{x_0\}}} |B_r|^{\lambda_1 + 1 - \frac{1}{p_1}} \|f_1\|_{L_{p_1}(B_{2r})}.$$

Then, by (4.9) and (4.10) we have

$$\begin{aligned}
&\left| T_\alpha^{(2)} [(b_1 - (b_1)_{B_r}) f_1^0, (b_2 - (b_2)_{B_r}) f_2^\infty] (x) \right| \\
&\leq \|b_1\|_{LC_{q_1, \lambda_1}^{\{x_0\}}} \|b_2\|_{LC_{q_2, \lambda_2}^{\{x_0\}}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 t^{n(\lambda_1 + \lambda_2) - \frac{n}{q} + n(\frac{1}{q_1} + \frac{1}{q_2}) - 1} \prod_{i=1}^2 \|f_i\|_{L_{p_i}(B_t)} dt.
\end{aligned}$$

Therefore,

$$\begin{aligned}
G_{24} &= \left\| T_{\alpha}^{(2)} \left[ (b_1 - (b_1)_{B_r}) f_1^0, (b_2 - (b_2)_{B_r}) f_2^{\infty} \right] \right\|_{L_q(B_r)} \\
&\lesssim \|b_1\|_{LC_{q_1, \lambda_1}^{\{x_0\}}} \|b_2\|_{LC_{q_2, \lambda_2}^{\{x_0\}}} r^{\frac{n}{q}} \\
&\quad \times \int_{2r}^{\infty} \left( 1 + \ln \frac{t}{r} \right)^2 t^{n(\lambda_1 + \lambda_2) - \frac{n}{q} + n(\frac{1}{q_1} + \frac{1}{q_2}) - 1} \prod_{i=1}^2 \|f_i\|_{L_{p_i}(B_t)} dt.
\end{aligned}$$

Putting estimates  $G_{21}$ ,  $G_{22}$ ,  $G_{23}$ ,  $G_{24}$  together, we get the desired conclusion

$$\begin{aligned}
G_2 &= \left\| T_{\alpha, (b_1, b_2)}^{(2)} (f_1^0, f_2^{\infty}) \right\|_{L_q(B_r)} \lesssim \|b_1\|_{LC_{q_1, \lambda_1}^{\{x_0\}}} \|b_2\|_{LC_{q_2, \lambda_2}^{\{x_0\}}} r^{\frac{n}{q}} \\
&\quad \times \int_{2r}^{\infty} \left( 1 + \ln \frac{t}{r} \right)^2 t^{n(\lambda_1 + \lambda_2) - \frac{n}{q} + n(\frac{1}{q_1} + \frac{1}{q_2}) - 1} \prod_{i=1}^2 \|f_i\|_{L_{p_i}(B_t)} dt.
\end{aligned}$$

(iii) Similarly, we have

$$\begin{aligned}
G_3 &= \left\| T_{\alpha, (b_1, b_2)}^{(2)} (f_1^{\infty}, f_2^0) \right\|_{L_q(B_r)} \lesssim \|b_1\|_{LC_{q_1, \lambda_1}^{\{x_0\}}} \|b_2\|_{LC_{q_2, \lambda_2}^{\{x_0\}}} r^{\frac{n}{q}} \\
&\quad \times \int_{2r}^{\infty} \left( 1 + \ln \frac{t}{r} \right)^2 t^{n(\lambda_1 + \lambda_2) - \frac{n}{q} + n(\frac{1}{q_1} + \frac{1}{q_2}) - 1} \prod_{i=1}^2 \|f_i\|_{L_{p_i}(B_t)} dt.
\end{aligned}$$

(iv) Finally, for  $G_4 = \left\| T_{\alpha, (b_1, b_2)}^{(2)} (f_1^{\infty}, f_2^{\infty}) \right\|_{L_q(B_r)}$ , we write

$$\begin{aligned}
G_4 &\lesssim \left\| (b_1 - (b_1)_{B_r}) (b_2 - (b_2)_{B_r}) T_{\alpha}^{(2)} (f_1^{\infty}, f_2^{\infty}) \right\|_{L_q(B_r)} \\
&\quad + \left\| (b_1 - (b_1)_{B_r}) T_{\alpha}^{(2)} [f_1^{\infty}, (b_2 - (b_2)_{B_r}) f_2^{\infty}] \right\|_{L_q(B_r)} \\
&\quad + \left\| (b_2 - (b_2)_{B_r}) T_{\alpha}^{(2)} [(b_1 - (b_1)_{B_r}) f_1^{\infty}, f_2^{\infty}] \right\|_{L_q(B_r)} \\
&\quad + \left\| T_{\alpha}^{(2)} [(b_1 - (b_1)_{B_r}) f_1^{\infty}, (b_2 - (b_2)_{B_r}) f_2^{\infty}] \right\|_{L_q(B_r)} \\
&\equiv G_{41} + G_{42} + G_{43} + G_{44}.
\end{aligned}$$

Let us estimate  $G_{41}$ ,  $G_{42}$ ,  $G_{43}$ ,  $G_{44}$  respectively.

Let  $1 < \tau < \infty$ , such that  $\frac{1}{q} = \left(\frac{1}{q_1} + \frac{1}{q_2}\right) + \frac{1}{\tau}$ . Then, by Hölder's inequality and noting that in (3.4)  $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha}{n}$ , we get

$$\begin{aligned}
G_{41} &= \left\| (b_1 - (b_1)_{B_r}) (b_2 - (b_2)_{B_r}) T_\alpha^{(2)}(f_1^\infty, f_2^\infty) \right\|_{L_q(B_r)} \\
&\lesssim \|b_1 - (b_1)_{B_r}\|_{L_{q_1}(B_r)} \|b_2 - (b_2)_{B_r}\|_{L_{q_2}(B_r)} \left\| T_\alpha^{(2)}(f_1^\infty, f_2^\infty) \right\|_{L_\tau(B_r)} \\
&\lesssim \|b_1\|_{L_{C_{q_1, \lambda_1}}^{\{x_0\}}} \|b_2\|_{L_{C_{q_2, \lambda_2}}^{\{x_0\}}} |B_r|^{(\lambda_1 + \lambda_2) + \left(\frac{1}{q_1} + \frac{1}{q_2}\right) + \frac{1}{\tau}} \\
&\quad \times \int_{2r}^{\infty} \|f_1\|_{L_{p_1}(B_t)} \|f_2\|_{L_{p_2}(B_t)} t^{-n\left(\frac{1}{p_1} + \frac{1}{p_2}\right) - 1 + \alpha} dt \\
&\lesssim \|b_1\|_{L_{C_{q_1, \lambda_1}}^{\{x_0\}}} \|b_2\|_{L_{C_{q_2, \lambda_2}}^{\{x_0\}}} r^{\frac{n}{q}} \\
&\quad \times \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 t^{n(\lambda_1 + \lambda_2) - \frac{n}{q} + n\left(\frac{1}{q_1} + \frac{1}{q_2}\right) - 1} \prod_{i=1}^2 \|f_i\|_{L_{p_i}(B_t)} dt.
\end{aligned}$$

Recalling the estimates used for  $G_{22}$ ,  $G_{23}$ ,  $G_{24}$  and also using the condition (1.4) with  $m = 2$ , we have

$$\begin{aligned}
&\left| T_\alpha^{(2)}[f_1^\infty, (b_2 - (b_2)_{B_r}) f_2^\infty](x) \right| \\
&\lesssim \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|b_2(y_2) - (b_2)_{B_r}| |f_1(y_1) \chi_{(B_{2r})^c}| |f_2(y_2) \chi_{(B_{2r})^c}|}{|(x_0 - y_1, x_0 - y_2)|^{2n - \alpha}} dy_1 dy_2 \\
&\lesssim \int_{(B_{2r})^c} \int_{(B_{2r})^c} \frac{|b_2(y_2) - (b_2)_{B_r}| |f_1(y_1)| |f_2(y_2)|}{|x_0 - y_1|^{n - \frac{\alpha}{2}} |x_0 - y_2|^{n - \frac{\alpha}{2}}} dy_1 dy_2 \\
&\lesssim \sum_{j=1}^{\infty} \int_{B_{2^{j+1}r} \setminus B_{2^j r}} \frac{|f_1(y_1)|}{|x_0 - y_1|^{n - \frac{\alpha}{2}}} dy_1 \int_{B_{2^{j+1}r} \setminus B_{2^j r}} \frac{|b_2(y_2) - (b_2)_{B_r}| |f_2(y_2)|}{|x_0 - y_2|^{n - \frac{\alpha}{2}}} dy_2 \\
&\lesssim \sum_{j=1}^{\infty} (2^j r)^{-2n + \alpha} \int_{B_{2^{j+1}r}} |f_1(y_1)| dy_1 \int_{B_{2^{j+1}r}} |b_2(y_2) - (b_2)_{B_r}| |f_2(y_2)| dy_2.
\end{aligned}$$

It's obvious that

$$(4.11) \quad \int_{B_{2^{j+1}r}} |f_1(y_1)| dy_1 \leq \|f_1\|_{L_{p_1}(B_{2^{j+1}r})} |B_{2^{j+1}r}|^{1 - \frac{1}{p_1}},$$



and using Hölder's inequality and by (4.7)

$$\begin{aligned}
& \int_{B_{2^{j+1}r}} |b_2(y_2) - (b_2)_{B_r}| |f_2(y_2)| dy_2 \\
& \leq \left\| b_2 - (b_2)_{B_{2^{j+1}r}} \right\|_{L_{q_2}(B_{2^{j+1}r})} \|f_2\|_{L_{p_2}(B_{2^{j+1}r})} |B_{2^{j+1}r}|^{1 - \left(\frac{1}{p_2} + \frac{1}{q_2}\right)} \\
& + \left| (b_2)_{B_{2^{j+1}r}} - (b_2)_{B_r} \right| \|f_2\|_{L_{p_2}(B_{2^{j+1}r})} |B_{2^{j+1}r}|^{1 - \frac{1}{p_2}} \\
& \lesssim \|b_2\|_{LC_{q_2, \lambda_2}^{\{x_0\}}} |B_{2^{j+1}r}|^{\frac{1}{q_2} + \lambda_2} \|f_2\|_{L_{p_2}(B_{2^{j+1}r})} |B_{2^{j+1}r}|^{1 - \left(\frac{1}{p_2} + \frac{1}{q_2}\right)} \\
& + \|b_2\|_{LC_{q_2, \lambda_2}^{\{x_0\}}} \left(1 + \ln \frac{2^{j+1}r}{r}\right) |B_{2^{j+1}r}|^{\lambda_2} \|f_2\|_{L_{p_2}(B_{2^{j+1}r})} |B_{2^{j+1}r}|^{1 - \frac{1}{p_2}} \\
(4.12) \quad & \lesssim \|b_2\|_{LC_{q_2, \lambda_2}^{\{x_0\}}} \left(1 + \ln \frac{2^{j+1}r}{r}\right)^2 |B_{2^{j+1}r}|^{\lambda_2 - \frac{1}{p_2} + 1} \|f_2\|_{L_{p_2}(B_{2^{j+1}r})}.
\end{aligned}$$

Hence, by (4.11) and (4.12), it follows that:

$$\begin{aligned}
& \left| T_\alpha^{(2)} [f_1^\infty, (b_2 - (b_2)_{B_r}) f_2^\infty] (x) \right| \\
& \lesssim \sum_{j=1}^{\infty} (2^j r)^{-2n+\alpha} \int_{B_{2^{j+1}r}} |f_1(y_1)| dy_1 \int_{B_{2^{j+1}r}} |b_2(y_2) - (b_2)_{B_r}| |f_2(y_2)| dy_2 \\
& \lesssim \|b_2\|_{LC_{q_2, \lambda_2}^{\{x_0\}}} \sum_{j=1}^{\infty} (2^j r)^{-2n+\alpha} \left(1 + \ln \frac{2^{j+1}r}{r}\right)^2 |B_{2^{j+1}r}|^{\lambda_2 - \left(\frac{1}{p_1} + \frac{1}{p_2}\right) + 2} \prod_{i=1}^2 \|f_i\|_{L_{p_i}(B_{2^{j+1}r})} \\
& \lesssim \|b_2\|_{LC_{q_2, \lambda_2}^{\{x_0\}}} \sum_{j=1}^{\infty} \int_{2^{j+1}r}^{2^{j+2}r} (2^{j+1}r)^{-2n+\alpha-1} \left(1 + \ln \frac{2^{j+1}r}{r}\right)^2 |B_{2^{j+1}r}|^{\lambda_2 - \left(\frac{1}{p_1} + \frac{1}{p_2}\right) + 2} \prod_{i=1}^2 \|f_i\|_{L_{p_i}(B_{2^{j+1}r})} dt \\
& \lesssim \|b_2\|_{LC_{q_2, \lambda_2}^{\{x_0\}}} \sum_{j=1}^{\infty} \int_{2^{j+1}r}^{2^{j+2}r} \left(1 + \ln \frac{2^{j+1}r}{r}\right)^2 |B_{2^{j+1}r}|^{\lambda_2 - \left(\frac{1}{p_1} + \frac{1}{p_2}\right) + 2} \prod_{i=1}^2 \|f_i\|_{L_{p_i}(B_{2^{j+1}r})} \frac{dt}{t^{2n-\alpha+1}} \\
& \lesssim \|b_2\|_{LC_{q_2, \lambda_2}^{\{x_0\}}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 |B_t|^{\lambda_2 - \left(\frac{1}{p_1} + \frac{1}{p_2}\right) + 2} \prod_{i=1}^2 \|f_i\|_{L_{p_i}(B_t)} \frac{dt}{t^{2n-\alpha+1}} \\
(4.13) \quad & \lesssim \|b_2\|_{LC_{q_2, \lambda_2}^{\{x_0\}}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 t^{n\lambda_2 - n\left(\frac{1}{p_1} + \frac{1}{p_2}\right) - 1 + \alpha} \prod_{i=1}^2 \|f_i\|_{L_{p_i}(B_t)} dt.
\end{aligned}$$

Let  $1 < \tau < \infty$ , such that  $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{\tau}$ . Then, by Hölder's inequality and (4.13), we obtain

$$\begin{aligned} G_{42} &= \left\| [(b_1 - (b_1)_{B_r})] T_\alpha^{(2)} [f_1^\infty, (b_2 - (b_2)_{B_r}) f_2^\infty] \right\|_{L_q(B_r)} \\ &\lesssim \|(b_1 - (b_1)_{B_r})\|_{L_{q_1}(B_r)} \left\| T_\alpha^{(2)} [f_1^\infty, (b_2 - (b_2)_B) f_2^\infty] \right\|_{L_\tau(B_r)} \\ &\lesssim \|b_1\|_{LC_{q_1, \lambda_1}^{\{x_0\}}} \|b_2\|_{LC_{q_2, \lambda_2}^{\{x_0\}}} r^{\frac{n}{q}} \\ &\quad \times \int_{2r}^\infty \left(1 + \ln \frac{t}{r}\right)^2 t^{n(\lambda_1 + \lambda_2) - \frac{n}{q} + n(\frac{1}{q_1} + \frac{1}{q_2}) - 1} \prod_{i=1}^2 \|f_i\|_{L_{p_i}(B_t)} dt. \end{aligned}$$

Similarly,  $G_{43}$  has the same estimate above, here we omit the details, thus the inequality

$$\begin{aligned} G_{43} &= \left\| (b_2 - (b_2)_{B_r}) T_\alpha^{(2)} [(b_1 - (b_1)_{B_r}) f_1^\infty, f_2^\infty] \right\|_{L_q(B_r)} \\ &\lesssim \|b_1\|_{LC_{q_1, \lambda_1}^{\{x_0\}}} \|b_2\|_{LC_{q_2, \lambda_2}^{\{x_0\}}} r^{\frac{n}{q}} \\ &\quad \times \int_{2r}^\infty \left(1 + \ln \frac{t}{r}\right)^2 t^{n(\lambda_1 + \lambda_2) - \frac{n}{q} + n(\frac{1}{q_1} + \frac{1}{q_2}) - 1} \prod_{i=1}^2 \|f_i\|_{L_{p_i}(B_t)} dt \end{aligned}$$

is valid.

Finally, to estimate  $G_{44}$ , similar to the estimate of (4.13), we have

$$\begin{aligned} &\left| T_\alpha^{(2)} [(b_1 - (b_2)_{B_r}) f_1^\infty, (b_2 - (b_2)_{B_r}) f_2^\infty] (x) \right| \\ &\lesssim \sum_{j=1}^\infty (2^j r)^{-2n+\alpha} \int_{B_{2^{j+1}r}} |b_1(y_1) - (b_1)_{B_r}| |f_1(y_1)| dy_1 \int_{B_{2^{j+1}r}} |b_2(y_2) - (b_2)_{B_r}| |f_2(y_2)| dy_2 \\ &\lesssim \|b_1\|_{LC_{q_1, \lambda_1}^{\{x_0\}}} \|b_2\|_{LC_{q_2, \lambda_2}^{\{x_0\}}} \int_{2r}^\infty \left(1 + \ln \frac{t}{r}\right)^2 t^{n(\lambda_1 + \lambda_2) - \frac{n}{q} + n(\frac{1}{q_1} + \frac{1}{q_2}) - 1} \prod_{i=1}^2 \|f_i\|_{L_{p_i}(B_t)} dt. \end{aligned}$$

Thus, we have

$$\begin{aligned} G_{44} &= \left\| T_\alpha^{(2)} [(b_1 - (b_1)_{B_r}) f_1^\infty, (b_2 - (b_2)_{B_r}) f_2^\infty] \right\|_{L_p(B_r)} \\ &\lesssim \|b_1\|_{LC_{q_1, \lambda_1}^{\{x_0\}}} \|b_2\|_{LC_{q_2, \lambda_2}^{\{x_0\}}} r^{\frac{n}{q}} \\ &\quad \times \int_{2r}^\infty \left(1 + \ln \frac{t}{r}\right)^2 t^{n(\lambda_1 + \lambda_2) - \frac{n}{q} + n(\frac{1}{q_1} + \frac{1}{q_2}) - 1} \prod_{i=1}^2 \|f_i\|_{L_{p_i}(B_t)} dt. \end{aligned}$$

By the estimates of  $G_{4j}$  above, where  $j = 1, 2, 3, 4$ , we know that

$$\begin{aligned} G_4 &= \left\| T_{\alpha, (b_1, b_2)}^{(2)} (f_1^\infty, f_2^\infty) \right\|_{L_q(B_r)} \lesssim \|b_1\|_{LC_{q_1, \lambda_1}^{\{x_0\}}} \|b_2\|_{LC_{q_2, \lambda_2}^{\{x_0\}}} r^{\frac{n}{q}} \\ &\quad \times \int_{2r}^\infty \left(1 + \ln \frac{t}{r}\right)^2 t^{n(\lambda_1 + \lambda_2) - \frac{n}{q} + n(\frac{1}{q_1} + \frac{1}{q_2}) - 1} \prod_{i=1}^2 \|f_i\|_{L_{p_i}(B_t)} dt. \end{aligned}$$

Recalling (4.6), and combining all the estimates for  $G_1, G_2, G_3, G_4$ , we get

$$\begin{aligned} \left\| T_{\alpha, (b_1, b_2)}^{(2)}(f_1, f_2) \right\|_{L_q(B_r)} &\lesssim \|b_1\|_{LC_{q_1, \lambda_1}^{\{x_0\}}} \|b_2\|_{LC_{q_2, \lambda_2}^{\{x_0\}}} r^{\frac{n}{q}} \\ &\times \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 t^{n(\lambda_1 + \lambda_2) - \frac{n}{q} + n\left(\frac{1}{q_1} + \frac{1}{q_2}\right) - 1} \prod_{i=1}^2 \|f_i\|_{L_{p_i}(B_t)} dt. \end{aligned}$$

Therefore, Theorem 5 is completely proved.  $\square$

Now we can give the following theorem, which is another our main result.

**Theorem 6.** (Our main result) Let  $x_0 \in \mathbb{R}^n$ ,  $0 < \alpha < mn$ , and  $1 \leq p_i < \frac{mn}{\alpha}$  for  $i = 1, \dots, m$  such that  $\frac{1}{q} = \sum_{i=1}^m \frac{1}{p_i} + \sum_{i=1}^m \frac{1}{q_i} - \frac{\alpha}{n}$  and  $\vec{b} \in LC_{q_i, \lambda_i}^{\{x_0\}}(\mathbb{R}^n)$  for  $0 \leq \lambda_i < \frac{1}{n}$ ,  $i = 1, \dots, m$ . Let also,  $T_{\alpha}^{(m)}$  ( $m \in \mathbb{N}$ ) be a multilinear operator satisfying condition (1.4), bounded from  $L_{p_1} \times \dots \times L_{p_m}$  into  $L_q$ . If functions  $\varphi, \varphi_i : \mathbb{R}^n \times (0, \infty) \rightarrow (0, \infty)$  ( $i = 1, \dots, m$ ) and  $(\varphi_1, \dots, \varphi_m, \varphi)$  satisfies the condition (4.14)

$$\int_r^{\infty} \left(1 + \ln \frac{t}{r}\right)^m t^{n\left(-\frac{1}{q} + \sum_{i=1}^m \lambda_i + \sum_{i=1}^m \frac{1}{q_i}\right) - 1} \operatorname{essinf}_{t < \tau < \infty} \prod_{i=1}^m \varphi_i(x_0, \tau) \tau^{\frac{n}{p_i}} dt \leq C \varphi(x_0, r),$$

where  $C$  does not depend on  $r$ .

Then the operator  $T_{\alpha, \vec{b}}^{(m)}$  ( $m \in \mathbb{N}$ ) is bounded from product space  $LM_{p_1, \varphi_1}^{\{x_0\}} \times \dots \times LM_{p_m, \varphi_m}^{\{x_0\}}$  to  $LM_{q, \varphi}^{\{x_0\}}$ . Moreover, we have

$$(4.15) \quad \left\| T_{\alpha, \vec{b}}^{(m)}(\vec{f}) \right\|_{LM_{q, \varphi}^{\{x_0\}}} \lesssim \prod_{i=1}^m \left\| \vec{b} \right\|_{LC_{q_i, \lambda_i}^{\{x_0\}}} \prod_{i=1}^m \|f_i\|_{LM_{p_i, \varphi_i}^{\{x_0\}}}.$$

*Proof.* Similar to the proof of Theorem 4, For  $1 < p_1, \dots, p_m < \infty$ , since  $(\varphi_1, \dots, \varphi_m, \varphi)$  satisfies (4.14) and by (3.8), we have

$$\begin{aligned} &\int_r^{\infty} \left(1 + \ln \frac{t}{r}\right)^m t^{n\left(-\frac{1}{q} + \sum_{i=1}^m \lambda_i + \sum_{i=1}^m \frac{1}{q_i}\right) - 1} \prod_{i=1}^m \|f_i\|_{L_{p_i}(B_t)} dt \\ &\leq \int_r^{\infty} \left(1 + \ln \frac{t}{r}\right)^m \frac{\prod_{i=1}^m \|f_i\|_{L_{p_i}(B_t)}}{\operatorname{essinf}_{t < \tau < \infty} \prod_{i=1}^m \varphi_i(x_0, \tau) \tau^{\frac{n}{p_i}}} \frac{\operatorname{essinf}_{t < \tau < \infty} \prod_{i=1}^m \varphi_i(x_0, \tau) \tau^{\frac{n}{p_i}}}{t^{n\left(\frac{1}{q} - \sum_{i=1}^m \lambda_i - \sum_{i=1}^m \frac{1}{q_i}\right)}} \frac{dt}{t} \\ &\leq C \prod_{i=1}^m \|f_i\|_{LM_{p_i, \varphi_i}^{\{x_0\}}} \int_r^{\infty} \left(1 + \ln \frac{t}{r}\right)^m \frac{\operatorname{essinf}_{t < \tau < \infty} \prod_{i=1}^m \varphi_i(x_0, \tau) \tau^{\frac{n}{p_i}}}{t^{n\left(\frac{1}{q} - \sum_{i=1}^m \lambda_i - \sum_{i=1}^m \frac{1}{q_i}\right) + 1}} dt \\ (4.16) \quad &\leq C \prod_{i=1}^m \|f_i\|_{LM_{p_i, \varphi_i}^{\{x_0\}}} \varphi(x_0, r). \end{aligned}$$

Then by (4.5) and (4.16), we get

$$\begin{aligned}
\left\| T_{\alpha, \vec{b}}^{(m)}(\vec{f}) \right\|_{LM_{q, \varphi}^{\{x_0\}}} &= \sup_{r>0} \varphi(x_0, r)^{-1} |B_r|^{-\frac{1}{q}} \left\| T_{\alpha, \vec{b}}^{(m)}(\vec{f}) \right\|_{L_q(B_r)} \\
&\lesssim \prod_{i=1}^m \left\| \vec{b} \right\|_{LC_{q_i, \lambda_i}^{\{x_0\}}} \sup_{r>0} \varphi(x_0, r)^{-1} \int_r^\infty \left( 1 + \ln \frac{t}{r} \right)^m t^{n \left( -\frac{1}{q} + \sum_{i=1}^m \lambda_i + \sum_{i=1}^m \frac{1}{q_i} \right) - 1} \prod_{i=1}^m \|f_i\|_{L_{p_i}(B_t)} dt \\
&\lesssim \prod_{i=1}^m \left\| \vec{b} \right\|_{LC_{q_i, \lambda_i}^{\{x_0\}}} \prod_{i=1}^m \|f_i\|_{LM_{p_i, \varphi_i}^{\{x_0\}}}.
\end{aligned}$$

Thus we obtain (4.15). Hence the proof is completed.  $\square$

For the multi-sublinear commutator of the multi-sublinear fractional maximal operator

$$M_{\alpha, \vec{b}}^{(m)}(\vec{f})(x) = \sup_{t>0} |B(x, t)|^{\frac{\alpha}{n}} \int_{B(x, t)} \frac{1}{|B(x, t)|} \prod_{i=1}^m [b_i(x) - b_i(y_i)] |f_i(y_i)| d\vec{y}$$

from Theorem 6 we get the following new results.

**Corollary 3.** *Let  $x_0 \in \mathbb{R}^n$ ,  $0 < \alpha < mn$ , and  $1 \leq p_i < \frac{mn}{\alpha}$  for  $i = 1, \dots, m$  such that  $\frac{1}{q} = \sum_{i=1}^m \frac{1}{p_i} + \sum_{i=1}^m \frac{1}{q_i} - \frac{\alpha}{n}$  and  $\vec{b} \in LC_{q_i, \lambda_i}^{\{x_0\}}(\mathbb{R}^n)$  for  $0 \leq \lambda_i < \frac{1}{n}$ ,  $i = 1, \dots, m$  and  $(\varphi_1, \dots, \varphi_m, \varphi)$  satisfies condition (4.14). Then, the operators  $M_{\alpha, \vec{b}}^{(m)}$  and  $\overline{T}_{\alpha, \vec{b}}^{(m)}$  ( $m \in \mathbb{N}$ ) are bounded from product space  $LM_{p_1, \varphi_1}^{\{x_0\}} \times \dots \times LM_{p_m, \varphi_m}^{\{x_0\}}$  to  $LM_{q, \varphi}^{\{x_0\}}$  for  $p_i > 1$  ( $i = 1, \dots, m$ ).*

**Remark 5.** *Note that, in the case of  $m = 1$  Theorem 6 and Corollary 3 have been proved in [24, 25].*

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